MILESTONE IN QUANTUM PHYSICS

1900 – Blackbody radiation Spectrum and Plank constant
1905 – Photoelectric effect
1913 – Bhor Atom Model
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1924 – De Broglie Wavelength
1927 – Principle of Indetermination of Heisenberg
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Appunti di Sara
Una rivisitazione del corso di Quantum Communication

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1900 – Blackbody radiation Spectrum and Plank constant

Max Planck introduced a revolutionary idea to solve the ultraviolet catastrophe in blackbody radiation. He proposed that electromagnetic energy could only be emitted or absorbed in discrete packets, or "quanta", proportional to the frequency: $E = h\nu$. This marked the birth of quantum theory and introduced Planck's constant h, a fundamental quantity in modern physics.

According to Plank, the radiation of a blackbody can be expressed as: $u(\nu, T) = \frac{8\pi\nu^3}{c^3}\frac{h\nu}{e^{\frac{h\nu}{kT}}-1}$

1905 – Photoelectric Effect

Albert Einstein explained the photoelectric effect by suggesting that light is made up of individual quanta called photons. He showed that light can eject electrons from a metal surface only if its frequency is above a certain threshold, regardless of intensity. This experiment confirmed the particle-like behavior of light and earned Einstein the Nobel Prize in Physics in 1921.

1913 – Bohr Atom Model

Niels Bohr proposed a new model of the atom in which electrons orbit the nucleus in quantized energy levels. Electrons could jump between these levels by absorbing or emitting photons with energy equal to the difference between levels. This model successfully explained the spectral lines of hydrogen and introduced the concept of quantized orbits.

1923 – Compton Effect

Arthur Compton demonstrated that X-rays scattered off electrons experience a shift in wavelength that depends on the scattering angle. This effect could only be explained if photons carried both energy and momentum, providing strong evidence for the particle nature of light and supporting the concept of photon-electron collisions.

1924 – De Broglie Wavelength

Louis de Broglie proposed that particles, like electrons, also have wave-like properties, with a wavelength given by $\lambda_B = \frac{h}{p}$. This hypothesis suggested that matter has both wave and particle nature, laying the foundation for wave mechanics and the later development of quantum mechanics.

1927 – Heisenberg's Uncertainty Principle

Werner Heisenberg formulated the Uncertainty Principle, stating that it is fundamentally impossible to know both the exact position and exact momentum of a particle at the same time. This principle reflects the inherent limitations of measurements in quantum mechanics and signifies a major departure from classical determinism.

1927 – Davisson-Germer Experiment

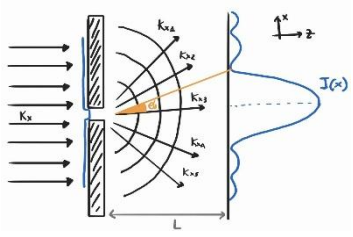
Clinton Davisson and Lester Germer experimentally confirmed electron diffraction by observing interference patterns when electrons were scattered off a crystal. This provided direct evidence of wave behavior in particles, confirming de Broglie's hypothesis and further establishing the wave-particle duality of matter.

Single Slit Experiment – Wave Nature of Light and Matter

The single slit diffraction experiment demonstrates how light behaves as a wave when it passes through a narrow opening. In this setup, a coherent light source, such as a laser, is directed at a single narrow slit (with width D). As the light passes through the slit, it spreads out and interferes with itself, forming a characteristic diffraction pattern on a screen placed behind the slit. This pattern consists of a central bright fringe (called the central maximum) flanked by several dimmer fringes (minima and secondary maxima) on both sides. The width of the slit and the wavelength of the light influence the spacing and intensity of these fringes. The experiment provides strong evidence for the wave nature of light and can be used to calculate the wavelength using the diffraction condition for minima: a·sin(θ) = mλ, where a is the slit width, θ the angle of the minima, λ the wavelength, and m an integer (excluding zero).

The wave at position z=0, corresponding to the slit can be expressed as:

$$\boldsymbol{\psi(x,0) = rect\left(\frac{x}{D}\right)}$$



It can be demonstrated that in "Fraunhofer diffraction regime" (when $\boldsymbol{L \gg \frac{D^2}{\lambda}}$), the spherical waves emerging from the slit can be approximated as plane waves. After that, we use the space Fourier Transform to move from $\boldsymbol{x}$ to $\boldsymbol{k_x}$

$$\boldsymbol{\psi(k_x,0) \propto \mathcal{F}\left(rect\left(\frac{x}{D}\right)\right) = D\,sinc\left(\frac{Dk_x}{2\pi}\right) = \left[k_x = \frac{2\pi}{\lambda}sen\theta\right] = D\,sinc\left(\frac{D\,sen\theta}{\lambda}\right)}$$

That gives a probability of:

$\boldsymbol{P(\theta) = |\psi(\theta)|^2 \propto sinc^2\left(\frac{D\,sen\theta}{\lambda}\right)}$ In Fraunhofer diffraction we can say that $\boldsymbol{sen\theta \sim \frac{x}{L}}$, resulting:

$$\boldsymbol{P(x) \propto sinc^2\left(\frac{Dx}{\lambda L}\right)}$$

Since the intensity is proportional to the amplitude of the wave squared, it will also be:

$$\boldsymbol{I(x) = I_0 sinc^2\left(\frac{Dx}{\lambda L}\right)}$$

Some useful parameters we can calculate to study deeper these phenomena are:

- Zeros of the sinc function:
  The intensity pattern reaches zero when: $x_m = m\frac{\lambda L}{D}$
  where $m \in Z \setminus \{0\}$. These positions correspond to destructive interference (dark fringes) in the diffraction pattern.
- Fresnel Number
  Defined as: $N_F = \left(\frac{D}{2}\right)^2 \frac{1}{\lambda L}$
  This number helps determine the diffraction regime:
  - $N_F \gg 1$: geometric optics
  - $N_F \sim 1$: Fresnel (near-field) diffraction
  - $N_F \ll 1$: Fraunhofer (far-field) diffraction

- Diffraction Angle
  The angle at which a given minimum occurs is given by: $sen\ \theta_m = \frac{m\lambda}{D}$

We can derive the same results also using the Indetermination Principle:

At position $z < 0$ (before the slit) we have:	$\Delta x = \Delta y = \Delta z = \infty$	$\Delta \bar{p} = 0$
At position $z = 0$ (in the slit)	$\Delta x = D$	$\Delta p_x = \frac{\hbar}{2D}$
Then moving with z:	$\frac{\Delta v_x}{v_z} = tg\theta_B = \frac{\lambda_B}{D}$	
Knowing that $\Delta v_x = \frac{\Delta p_x}{m_e} = \frac{\hbar}{2Dm_e}$	$\frac{\Delta v_x}{v_z} = tg\theta_B = \frac{\lambda_B}{D} = \frac{\hbar}{2Dm_e v_z}$	

Double Slit Experiment – Wave and Particle Nature of Light and Matter

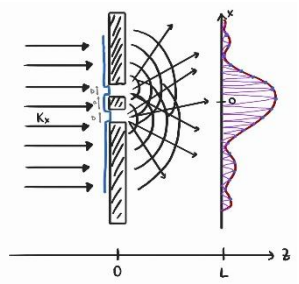
Double slit diffraction is a fundamental experiment in physics that demonstrates the wave nature of light and other particles. When a coherent light source, such as a laser, passes through two closely spaced slits, it produces an interference pattern of bright and dark fringes on a screen, due to the constructive and destructive interference of the light waves. However, if a detector is placed to observe which slit the particle goes through, the interference pattern disappears, and a particle-like behavior is observed instead. This surprising result highlights the wave-particle duality of matter and light, a central concept in quantum mechanics. The experiment, first performed by Thomas Young in the early 19th century, remains one of the most striking illustrations of quantum behavior.

The wave at position z=0, corresponding to the slit can be expressed as:

$$\psi(x,0) \propto \left[rect\left(\frac{x-\frac{a}{2}}{D}\right) + rect\left(\frac{x+\frac{a}{2}}{D}\right)\right]$$

Each rect, if the width of the hole is small enough, can be seen as two deltas and each of them is a source of spherical wave.

$\psi(k_x, z) \propto \frac{e^{ik_x r_a}}{r_a} + \frac{e^{ik_x r_b}}{r_b}$ $\quad for\ 0 < z < L$ (the wavefunction must be unitary, that's why we normalized)



In general, if a wavefunction is expressed as $\psi = \psi_1 + \psi_2$, the probability to find the particle in a certain position is given by:

$$P = |\psi|^2 = [\psi_1^* + \psi_2^*][\psi_1 + \psi_2] = |\psi_1|^2 + |\psi_2|^2 + 2\,Re\{\psi_1\psi_2\} = P_1 + P_2 + 2\sqrt{P_1 P_2}\,cos\phi$$

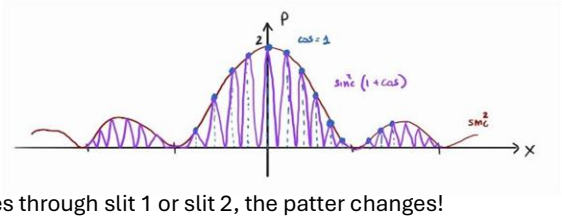
So, the probability of finding the particle in a certain position is given to the probability distribution given from the first slit + the probability distribution given by the second slit + an interference term.

In paraxial approximation (near x=0) we can say that $\psi \propto \frac{e^{ik_x r_a}}{r_a} + \frac{e^{ik_x r_b}}{r_b} \sim \frac{e^{ik_x r_a}}{L} + \frac{e^{ik_x r_b}}{L}$, each rect will introduce a sinc in the pattern, but since the two rects are very close, we can assume that the two sinc are identical as the single slit experiment $P_1(x) = P_2(x) = sinc^2\left(\frac{Dx}{\lambda L}\right)$

$$\begin{cases} P = |\psi_1|^2 + |\psi_2|^2 + 2\,Re\{\psi_1\psi_2\} = |\psi_1|^2 + |\psi_2|^2 + 2\sqrt{\psi_1\psi_2}\cos\phi(x) \propto sinc^2\left(\frac{Dx}{\lambda L}\right)\left(1 + \cos\left(k_x\frac{ax}{L}\right)\right) \\ \phi(x) = \arg(\psi_1(x)) - \arg(\psi_2(x)) = \arg\left(\frac{e^{ik_x r_a}}{L}\right) - \arg\left(\frac{e^{ik_x r_b}}{L}\right) = k_x(r_a(x) - r_b(x)) \rightarrow \ k_x\frac{ax}{L} \\ r_a(x) = \sqrt{\left(x+\frac{a}{2}\right)^2 + L^2} = L\sqrt{1 + \left(\frac{x+\frac{a}{2}}{L}\right)^2} = \left[\sqrt{1+x} \approx 1 + \frac{1}{2}x\right] = L\left[1 + \frac{1}{2}\left(\frac{x+\frac{a}{2}}{L}\right)^2\right] = L + \frac{\left(x+\frac{a}{2}\right)^2}{2L} \\ r_a(x) = \sqrt{\left(x-\frac{a}{2}\right)^2 + L^2} = \ L + \frac{\left(x-\frac{a}{2}\right)^2}{2L} \end{cases}$$

After all, we find that:

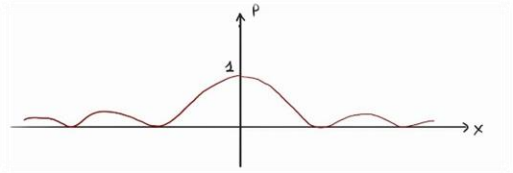
$$\boldsymbol{P_{out} = |\psi_1 + \psi_2|^2 \propto sinc^2\left(\frac{Dx}{\lambda L}\right)\left(1 + \cos\left(\frac{2\pi}{\lambda}\frac{xa}{L}\right)\right)}$$



Instead, if we introduce a path detector that measures whether the particle passes through slit 1 or slit 2, the patter changes!
This happens because measuring the particle 3forces the particle to be there, or, in other words, taking a measurement changes the quantum state of the particle.

In a double slit experiment with a path detector, we find:

$$\boldsymbol{P_{out} = |\psi_1|^2 + |\psi_2|^2 \propto sinc^2\left(\frac{Dx}{\lambda L}\right)}$$



An additional concept that is useful to study this, is the "**visibility of interference**". Visibility quantifies how clearly the interference fringes appear and is defined as:

$$V = \frac{P_{max} - P_{min}}{P_{max} + P_{min}}$$

A visibility of 1 indicates perfect interference (clear, high-contrast fringes) (wave nature), while a visibility of 0 means no interference is observed (particle nature). In the context of quantum mechanics, the visibility is often used to measure the coherence of the system and the degree to which "which-path" information destroys the interference, reflecting the balance between wave-like and particle-like behavior.
In a complementary way, we define the parameter of "**distinguishability**":

$$\eta = 1 - V$$

η of 1 means no interference while a η of 0 means full interference.

In this way we can write the probability of finding a particle in a certain position in a single equation, both for the wave nature and when a path detector is added:

$$\boldsymbol{P_{out} \propto sinc^2\left(\frac{Dx}{\lambda L}\right)\left(1 + (1-\eta)\cos\left(\frac{2\pi}{\lambda}\frac{xa}{L}\right)\right)}$$

QUANTUM STATES AND OPERATORS

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1) The mathematical model for quantum physics (Hilbert Space, Inner Products, Operators and so on)

Why and How modeling Quantum Mechanics with Linear Algebra.
https://www.youtube.com/watch?v=3nvbBEzfmE8

In classical physics we know that physical quantities are:
- Single-valued
- Continuous

And given that, we use continuous functions to study these phenomena (s(t), v(t), a(t), T(t), etc.)

In quantum physics out physical quantities are:
- Random but probabilistic
- Discrete (quantization of energy)

For example, when measuring the energy emitted by irradiated hydrogen, one would typically expect the measured energy to correspond to the energy gap. However, the presence of defects can alter this expectation. In rarer cases, different energy values may be observed, depending on the energy levels introduced by such defects.
Given that, we clearly understand that a continuous function cannot represent our model.

We can represent each possible value (of energy for example) like $A, B, C, D, \dots$, so our particle in these specific cases can be represented by mathematical objects, vectors, like $M_A, M_B, M_C, M_D, \dots$
Then we must create a mathematical object that contains all possible vectors because before the measurement we do not know the output = $M_A \cdot M_B \cdot M_C \cdot M_D \dots$, a space that may contains all the possible vectors.
Each mathematical object must carry a weight to indicate its probability: $a_A M_A \cdot a_B M_B \cdot a_C M_C \cdot a_D M_D \dots$ that indicates a linear combination, so we can deduct that a particle is a linear combination of all outcome-possibilities with each possibility represented by a vector. In this way we described the random but probabilistic feature.
To describe the discrete feature of a physical quantity we use a matrix. When we are interested in calculating the physical quantity of a particle, we will have to submit our particle (linear combination) to the matrix (physical quantity).

Our vectors (M_A, M_B, M_C, M_D) are called eigenvectors, the values (A, B, C, D) are called eigenvalues and our matrix is an operator.

Vectors, Ket and Wavefunctions
https://www.youtube.com/watch?v=DBZ_hCmj8Mk

We start defining what a vector space is: a vector space is a set of objects, called vectors, that satisfy these rules:

$\vec{v} + \vec{u}$ gives a vector as an output;	$a(b\vec{u}) = (ab)\vec{u}$
$a\vec{v}$ gives a vector as an output;	$1\vec{u} = \vec{u}$
$(\vec{v} + \vec{u}) + \vec{w} = \vec{v} + (\vec{u} + \vec{w})$	$a(\vec{v} + \vec{u}) = a\vec{v} + a\vec{u}$
$\vec{v} + \vec{u} = \vec{u} + \vec{v}$	$(a + b)\vec{u} = a\vec{u} + b\vec{u}$

Keeping in mind that a vector is not only the traditional arrow but any mathematical object that follows these rules.

In quantum mechanics, we define a particle as a vector in the vector space. We call this vector a quantum state $\vec{\psi}$. From this quantum state we should be able to extract all the information like energy and position and all the relative weights.
A quantum state, or in general a vector, can be represent both like $\vec{\psi}$ and $|\psi\rangle$.
A quantum state that represents all the possible values of energy that can be measured by irradiated hydrogen can be written as:

$$|\phi\rangle = c_1|E_1\rangle + c_2|E_2\rangle + c_3|E_3\rangle$$

and can be seen as a superposition of all quantum states that our particles can be seen.
A quantum state that represents all the possible values of momentum, in a similar way, can be written as:

$$|\phi\rangle = c_1|P_1\rangle + c_2|P_2\rangle + c_3|P_3\rangle$$

The same quantum state $|\phi\rangle$ can both be represented as a linear combination of energy or momentum or position …

Now we should consider that energy is quantized (thanks Plank), but some other physical quantities, like position, are continuous.
Until now we have described the quantum state as a sum of all possible values and now, we should pass to the integral form:

$|\phi\rangle = c_1|E_1\rangle + c_2|E_2\rangle + c_3|E_3\rangle = \sum_i c_i|E_i\rangle$ $\qquad$ $|\phi\rangle = \int_{-\infty}^{+\infty} dx\, c(x)|x\rangle = \int_{-\infty}^{+\infty} dx\, \phi(x)|x\rangle$

When we represent the quantum state as the integral of all the possible positions, we write our continuous weight coefficients $c(x)$ as a position wavefunction because that what it is!
In quantum mechanics a wavefunction is a mathematical function that encapsulates the quantum state of a system. It provides the probability amplitude for the outcomes of measurements of physical quantities.

Hilbert Space and the concept of infinity
https://www.youtube.com/watch?v=_kJUUxjJ_FY&t=24s

The dimensions of a vector space is the number of (independent) vectors to form that space, for example, if we can describe our quantum state as: $|\phi\rangle = c_1|E_1\rangle + c_2|E_2\rangle + c_3|E_3\rangle$ it means that the dimension of the space that represents our quantum state is 3.
In some cases, we may have that a quantum state can be represented as a superposition of infinite number of vectors leading to an infinite dimension of the space.
The concept of infinite may be dangerous because a sum of infinite vectors may lead to a quantum state that is outside the vector space!
The general vector space as a representation of our mathematical object is no longer sufficient. We would like to add the constraint: Every convergent sum of vectors must converge to an element inside the vector space: $\sum_i^\infty |E_i\rangle \rightarrow |\phi\rangle$, and with that, we described a Hilbert Space!
A Hilbert Space is a vector space with an inner product (scalar product) that is Cauchy complete, in simple, every convergent sequence of vectors (partial or infinite linear combination) converges to an element inside the vector space. Every quantum state represents a ray of an Hilbert Space.

Focus on the Inner Product
https://www.youtube.com/watch?v=3N2vN76E-QA

We first review the importance of the scalar product in Euclidean Space $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$:

- When the scalar product gives 0 (with non-null vectors), the two vectors are defined as orthogonal.
- We define the length of a vector, following Pythagoras, as $|\vec{u}| = \sqrt{u_x^2 \cdot u_y^2} = \sqrt{\vec{u} \cdot \vec{u}}$

In our Hilbert Space, since we use a fancier way to write vectors, we define our scalar product as $InProd(|\psi\rangle, |\phi\rangle)$ or simply $\langle\psi|\phi\rangle$
All the properties of the vectors are applied also to our bras and kets, so:

$|\phi\rangle + |\psi\rangle = |\phi + \psi\rangle$ $\quad\quad$ $\langle\phi| + \langle\psi| = \langle\phi + \psi|$
$\langle\psi|a\phi\rangle = a\,\langle\psi|\phi\rangle$ $\quad\quad$ $\langle a\,\psi|\phi\rangle = a^*\,\langle\psi|\phi\rangle$
$\langle\psi|\phi + \zeta\rangle = \langle\psi|\phi\rangle + \langle\psi|\zeta\rangle$ $\quad\quad$ $\langle\psi + \phi|\zeta\rangle = \langle\psi|\zeta\rangle + \langle\phi|\zeta\rangle$

So, the kets are linear while the bras are antilinear.
In particular, $\langle a\,\psi|\,a\,\psi\rangle = a\,a^*\,\langle\psi|\psi\rangle = 1$, $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$ and we also ensure that the only the zero vector has zero length.
As the Euclidean space, we define the length as: $\sqrt{\langle\psi|\psi\rangle}$ and the definition of orthogonality as $\langle\psi|\phi\rangle = 0$
We say that a vector is normalized if $||\psi||^2 = 1$ and to normalize a vector: $|\psi'\rangle = \frac{|\psi\rangle}{\sqrt{\langle\psi|\psi\rangle}}$

Inner Product as the method to extract information of a quantum state
https://www.youtube.com/watch?v=nDa3cqFk80o&t=414s

Now we will see the reason we are interested in the inner product: the inner product has the capacity to extract a value of the wavefunction, so a single probability of a quantum state.
First of all, we need to introduce what an orthonormal basis is: $\{|E_i\rangle\}$ is an orthonormal basis if $\langle E_i|E_i\rangle = 1$ and $\langle E_i|E_j\rangle = 0$.
In the discrete world we can indicate an orthonormal basis with the Kronecker delta δ_{ij} that it's equal to 1 when $i = j$ and 0 otherwise.
In the continuous world we use a Dirac delta to indicate an orthonormal basis $\delta(x)$
Our definition of Dirac delta is not just the spike definition, but we define it as any function that satisfies $f(c) = \int dx\, f(x)|\delta(c - x)$.

Now we can start with a quantum state defined as a superposition of finite quantum states and see how we can extract a single coefficient through the inner product:
$\langle E_2|\phi\rangle = \langle E_2|(\sum_i c_i|E_i\rangle) = \sum_i c_i\langle E_2|E_i\rangle$
Taking to account that E_2, E_i are orthonormal basis (we would just need another coefficient outside) we can write:
$\langle E_2|\phi\rangle = \langle E_2|(\sum_i c_i|E_i\rangle) = \sum_i c_i\langle E_2|E_i\rangle = \sum_i c_i\,\delta_{2i} = c_2$
For two general quantum states we have:
$\langle\psi|\phi\rangle = (\sum_i\langle a_i E_i|)\big(\sum_j |b_j E_j\rangle\big) = \sum_i\sum_j\langle a_i E_i|b_j E_j\rangle = \sum_i\sum_j a_i^* b_j\,\langle E_i|E_j\rangle = \sum_i\sum_j a_i^* b_j\,\delta_{ij} = \sum_i a_i^* b_i$ $\quad$ that is the scalar product!!

In a similar way, for a quantum state defined as a superposition of infinite
$\langle 2.71|\phi\rangle = \langle 2.71\,|\int dx\, c(x)|x\rangle = \int dx\, c(x)\,\langle 2.71|x\rangle = \int dx\, c(x)\,\delta(2.71 - x) = c(2.71)$
For two general quantum states we have:
$\langle\psi|\phi\rangle = \big(\int dx\,\psi(x)\langle x|\big)\big(\int dy\,\phi(y)|y\rangle\big) = \int\int dx\,dy\,\psi(x)^*\phi(y)\,\langle x|y\rangle = \int\Big(\int dx\;\psi(x)^*\phi(y)\,\delta(x - y)\Big)dy =$
$\int dx\;\psi(x)^*\phi(x)\,\delta(x - x) = \int dx\;\psi(x)^*\phi(x)$

Focus on Bra
https://www.youtube.com/watch?v=lRR-qgjaKlg

We saw that with the inner product, the bra helps us to extract information from the quantum state represented by ket. Let's define what bra really is in math language. A bra is a linear functional living in the Hilber Dual Space. (Riesz Representation Theorem)
An example can be in $\mathbb{R}^2$, with a simple linear functional L_x that has the role of taking the first component of a vector like:
$L_x * \begin{bmatrix} -3 \\ 7 \end{bmatrix} = -3$ and so on. It can be calculated that in this case $L_x = [1 \quad 0]$ and it's a row matrix
The definition of a linear functional is any linear map that goes from the vector space to a scalar number.
All the linear functionals (in each $\mathbb{R}^n$) form a vector space that is called dual space which definition is: given a vector space V, the dual space V^* is the vector space of all linear functionals in V. In fact, the bra lives in the Hilbert Dual Space $\langle\psi| \sqsubseteq \mathcal{H}^*$.

Observables Operators
https://www.youtube.com/watch?v=ANLRQ7X6h5A

Until now we saw that particles can be represented by quantum state with the ket notation and to extract information on the probability of having a deterministic value of a physical quantity we use linear functionals represented with the bra notation.
Now we must discuss how to represent physical quantity in a mathematical way. First of all, in quantum mechanics we do not talk about "physical quantity" but rather of "observable" to indicate any physical quantity that can be measured out of a particle (x, p, E, L).
When we introduced the new model for quantum mechanics, we stated that an "observable" is a matrix that acts on a particle (quantum state), returning all the possible values that the particle can assume. This matrix is a linear operator.
A linear operator is a map on a vector space that preserves the linear structure of the space, so it satisfies these proprieties:
$\widehat{M}(|\psi\rangle + |\phi\rangle) = \widehat{M}(|\psi\rangle) + \widehat{M}(|\phi\rangle)$ and $\widehat{M}(c|\phi\rangle) = c\widehat{M}(|\phi\rangle)$
Now we should understand how to build this matrix and how the particle submitted to the matrix gives different outputs.
We can start measuring multiple times a certain physical quantity and obtain all the possible values. Taking the energy output, we may measure:
$E_1 = 0.44\ eV$ that represents the first possible quantum state $|E_1\rangle$
$E_2 = 0.78\ eV$ that represents the second possible quantum state $|E_2\rangle$
$E_3 = 1.02\ eV$ that represents the third possible quantum state $|E_3\rangle$
$E_4 = 0.82\ eV$ that represents the fourth possible quantum state $|E_4\rangle$
And overall, we have that $\hat{E}|\psi\rangle = a_1|E_1\rangle + a_2|E_2\rangle + a_3|E_3\rangle + a_4|E_4\rangle$, $\hat{L}|\psi\rangle = a_1|L_1\rangle + a_2|L_2\rangle + a_3|L_3\rangle$, etc.
The column on the left are eigenvalues, while the column on the right are eigenstates and an eigenvector $|\psi\rangle$ is represented by the linear combination of eigenstates.

Now we can derive some properties that observables should have:
1) Observables need to have real eigenvalues
2) Observables' eigenstates must span the entire vector space (they are the combination of orthonormal basis of the space and so any quantum state can be written as a linear combination of eigenstates)
3) Eigenstates must be mutually orthogonal

So, as observable's eigenstates must form an orthonormal eigenbasis, in fact operators in quantum mechanics are Hermitian operators.

Born Rule
https://www.youtube.com/watch?v=PZUZgOUOOIU

We found that $\hat{E}|\psi\rangle = a_1|E_1\rangle + a_2|E_2\rangle + a_3|E_3\rangle + a_4|E_4\rangle$ and we said that through the inner product we can extract a certain coefficient. Now we focus on how to find the probability coefficient that somewhat is correlated with our coefficients $a_1, a_2, a_3, \ldots$
From an intuitive perspective, considering a 2-dimensional space, with $\widehat{u_x}$ *and* $\widehat{u_y}$ as orthonormal basis, a vector that is a linear combination of the two has more probability of falling in a particular eigenstate $|E_x\rangle$ if the vector is very close to $\widehat{u_x}$; in other words, if a projection of a general vector on an eigenvector is high ($\vec{v}\cos\alpha \sim \vec{v}$) than the probability of having the eigenstate relative of that eigenvector is high. And remember that the projection of a vector on an eigenvector is the inner product!! Now we prove that a certain coefficient contains information of the probability of the relative eigenstate of the coefficient.
$\langle\widehat{u_x}|\psi\rangle = \langle\widehat{u_x}|\sum_i c_i|E_i\rangle = \sum_i c_i\langle\widehat{u_x}|E_i\rangle = c_x\,\delta_{xx} + c_y\delta_{xy} = c_x$
(we take the dimension of parallel component to the direction we are interested in)
The actual probability is defined by the magnitude squared of the coefficient $P(E = E_1) = |c_1|^2$ (the coefficient could be complex but the probability must be real and the sum of all the probabilities must be 1) (this result can be achieved resolving a differential equation. If interested, watch the video or check the Gleason's Theorem). Overall, the Born rule says (considering all the eigenvectors normalized):

$$P(E = E_i) = |c_i|^2 = |\langle E_i|\psi\rangle|^2 \; and \sum p_i = 1$$

In the same way, for a continuous physical quantity:

$$P(x) = |\psi(x)|^2 \; and \sum p_i = 1$$

In case our eigenvectors aren't normalized, a general expression is: $p_i = FIDELTY\ (\psi, E_i) = \frac{|\langle\psi|E_i\rangle|^2}{\langle\psi|\psi\rangle\langle E_i|E_i\rangle}$

The Hermitian Operator
https://www.youtube.com/watch?v=da1rH0Hq62Q

We have seen that operators work like $\widehat{M}|\psi\rangle$ and return a vector. When we have an inner product with an operator inside $\langle\phi|\widehat{M}|\psi\rangle$ that is the same as writing $\langle\phi|\widehat{M}\psi\rangle$, the result is the projection of $|\widehat{M}\psi\rangle$ on $\langle\phi|$. Intuitively we may also think that should exist an operator $\widehat{N}$ that applies on $\langle\phi|$ so that $\langle\widehat{N}\phi| = |\widehat{M}\psi\rangle$. This operator $\widehat{N}$ is called Hermitian adjoint and it's indicated $\widehat{M^{\dagger}}$ (Dagger).
Hermitian adjoint have the following properties:

$(\widehat{M^{\dagger}})^{\dagger} = \widehat{M}$ $\quad$ $(\hat{A}+\hat{B})^{\dagger} = \hat{A}^{\dagger} + \hat{B}^{\dagger}$ $\quad$ $(\hat{A}\hat{B})^{\dagger} = \hat{B}^{\dagger}\hat{A}^{\dagger}$

From these we can derive some other proprieties: (c is a scalar)

$c^{\dagger} = c^{*}$ $\quad$ $\langle\phi|\psi\rangle^{\dagger} = |\psi\rangle^{\dagger}\langle\phi|^{\dagger} = \langle\psi|\phi\rangle$

$\langle\phi|\psi\rangle^{\dagger} = \langle\phi|\psi\rangle^{*} = \langle\psi|\phi\rangle$ $\quad$ $|\psi\rangle^{\dagger} = \langle\psi| \; and \; \langle\phi|^{\dagger} = |\phi\rangle$

We introduced Hermitian adjoint because it will be interesting calculating the Hermitian adjoint of an observable:
We have said that observables are represented by linear operators whose eigenvectors represent definite states and eigenvalues represent the corresponding measured value. It can be proved (if interested, watch the video) that, for an observable:

$$\hat{E}^{\dagger} = \hat{E}$$

Operators that are their own Hermitian adjoints are called Hermitian Operators, and so $\langle\phi|\hat{E}\psi\rangle = \langle\hat{E}\phi|\psi\rangle$

The Observable Commutate
https://www.youtube.com/watch?v=-pRk9HNh7os

An important mathematical tool to study operators is the commutator.
We say that two operators commute if $\hat{A}\hat{B} = \hat{B}\hat{A}$ (usually it is not true for two general operators) and this property helps us to simplify many expressions.
Given two operators $\hat{A}$ and $\hat{B}$, their commutator is represented by $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. So, if two operators commute, their commutator is equal to zero and vice versa. In a more general case, we can say $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = \hat{C}$.
We use this property so that we can always slip products (that simplify our life) if we remember to add the commutator $\hat{A}\hat{B} = \hat{B}\hat{A} + \hat{C}$.
It can be proved that (watch video if interested) if two observables $\hat{A}$ and $\hat{B}$ commute, then they share a simultaneous eigenbasis $\{|\widehat{A_i}\widehat{B_i}\rangle\}$.
This is important because it tells us that we can make two measurements at the same time and the quantum state will collapse into a definite eigenvector for both the observables:
$[\hat{p},\hat{E}] = 0 \;\; then \;\; it \; exists \; the \; eigenstate \; |p_1 E_1\rangle$, … , so if I measure p_1 for sure I also know that the energy will be E_1

Instead, if two operators do not commutate, they don't have a simultaneous eigenbasis and there exists eigenstates of one observable that are always in a superposition of the other's eigenstate.
$[\hat{p},\hat{x}] \neq 0 \;\; then \;\; it \; exists \; the \; eigenstate \; |p_1\rangle = c_1|E_1\rangle + c_2|E_2\rangle + \cdots$, so, if I measure p_1 , the energy could assume all the possible values and we are not able to measure two physical quantities at the same time because the certainty of one will determine the uncertainty of the other.

NB: The position operator and the momentum operator do not commute indeed, and this is the origin of Heisenberg uncertainty principle.

The Unitary Operators
https://www.youtube.com/watch?v=dD-oYfhSKhg

Another important class of operators are the unitary operators. Unitary operators are those operators that satisfy: $\langle\widehat{U}\psi|\;\widehat{U}\phi\rangle = \langle\psi|\;\phi\rangle$
Another important property is $\langle\widehat{U}\psi|\;\widehat{U}\psi\rangle = \langle\lambda\psi|\;\lambda\psi\rangle = \lambda\lambda^{*}\langle\psi|\;\psi\rangle = |\lambda|^{2}\langle\psi|\;\psi\rangle = \langle\psi|\;\psi\rangle \; so \; |\lambda|^{2} = 1$, so the eigenvalues of a unitary operators must have magnitude one.
Unitary operators are useful since they maintain the inner product and so applying them won't change the probability of a certain eigenvalue.

Lagrangian: quantum state and classical state
https://www.youtube.com/watch?v=Uorwy0BQGU

We have understood that a quantum state is a powerful mathematical object that contains all the information of a particle. In classical mechanics it exists a mathematical object that carries more than one information, this is called the Lagrangian: $\mathcal{L}(t, x(t), \dot{x}(t))$ and it represent the "classical state" of a particle.

$\mathcal{L} = \mathcal{E}_k - \mathcal{E}_p = \frac{1}{2} m\, \dot{x}(t) - V(x) = \frac{p^2}{2m} - V(x)$

There's a famous equation that the Lagrangian always should satisfy to be a candidate of the optimal solution: the Euler-Lagrange equation $\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$

Let's solve the Euler-Lagrange equation with the Lagrangian defined as the difference between the kinetic and the potential energy:

$-\frac{d}{dx} V(x) = \frac{d}{dt}\,(m\dot{x}) \rightarrow F = m\,\ddot{x}$ that is the second Newton Law!

$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\,(m\dot{x}) \rightarrow \frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\,p$ so, a variation of position of a particle can be calculated by looking at the change in its momentum

Other laws we can derive are:

$\frac{\partial \mathcal{L}}{\partial p} = \frac{p}{m} = \frac{d}{dt}\,x$ so, a variation of momentum of a particle can be calculated by looking at the change in its position

$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial t} + \frac{\partial \mathcal{L}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \mathcal{L}}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial t} = \frac{\partial \mathcal{L}}{\partial t} + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\dot{x} + \frac{\partial \mathcal{L}}{\partial \dot{x}}\left(\frac{\partial \dot{x}}{\partial t}\right) = \frac{\partial \mathcal{L}}{\partial t} + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\,\dot{x}\right) \rightarrow \frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial t} + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\,\dot{x}\right) \rightarrow \frac{\partial \mathcal{L}}{\partial t} = \frac{d}{dt}\left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{x}}\,\dot{x}\right)$

$\frac{\partial \mathcal{L}}{\partial t} = \frac{d}{dt}\,(\mathcal{L} - m\dot{x}\dot{x}) = \frac{d}{dt}\left(\frac{1}{2} m\,\dot{x}(t) - V(x) - m\dot{x}^2\right) \rightarrow \frac{\partial \mathcal{L}}{\partial t} = -\frac{d}{dt}E$ a variation of the particle in time equals a variation of energy

$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\,L$

We have found that:

$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\,p$ $\quad$ $\frac{\partial \mathcal{L}}{\partial p} = \frac{d}{dt}\,x$ $\quad$ $\frac{\partial \mathcal{L}}{\partial t} = -\frac{d}{dt}E$ $\quad$ $\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\,L$

We said that our Lagrangian is the "classical state" of a particle. In the quantum world we have:

$\frac{\partial}{\partial x}|\psi\rangle \propto \hat{p}$ $\quad$ $\frac{\partial}{\partial p}|\psi\rangle \propto \hat{x}$ $\quad$ $\frac{\partial}{\partial t}|\psi\rangle \propto \hat{E}$ $\quad$ $\frac{\partial}{\partial \theta}|\psi\rangle \propto \hat{L}$

The General Schrodinger Equation
https://www.youtube.com/watch?v=KmFG_QNSZzA

We want to focus on this equation $\frac{\partial}{\partial t}|\psi\rangle \propto -\frac{d}{dt}\hat{E}$ to derive the famous Schrodinger Equation $i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle$.
First, we want to discuss how we represent the time evolution in quantum physics. We can think of it as a time evolution operator, like:
$\hat{U}(t)|\psi\rangle = |\psi(t)\rangle$
The properties of this operator are:
$\hat{U}(0) = \hat{I}$ (identity operator)
$\hat{U}^{-1}(t)$ exist, so we can move forward and backward in time and find the same value again, so time evolution is reversable.
$\langle \hat{U}(t)\psi | \hat{U}(t)\psi \rangle = \langle \psi | \psi \rangle = 1$ so, the time evolution operator is a unitary operator, so probability is conserved through time.

Now let's consider the time evolution operator that actions in a small chunk of time dt and use Taylor:

$\hat{U}(dt) = \hat{U}(0) + \dot{\hat{U}}(0)dt + \hat{O}(dt^2)$

$\hat{U}(dt)|\psi\rangle = |\psi\rangle + \dot{\hat{U}}(0)dt|\psi\rangle + \hat{O}(dt^2)|\psi\rangle \rightarrow |\psi(dt)\rangle = |\psi\rangle + \dot{\hat{U}}(0)dt|\psi\rangle + \hat{O}(dt^2)|\psi\rangle \rightarrow |\psi(dt)\rangle - |\psi\rangle = \dot{\hat{U}}(0)dt|\psi\rangle + \hat{O}(dt^2)|\psi\rangle$

$\frac{|\psi(dt)\rangle - |\psi\rangle}{dt} = \frac{\dot{\hat{U}}(0)dt|\psi\rangle + \hat{O}(dt^2)|\psi\rangle}{dt} \rightarrow lim_{dt\to 0}\ \frac{d}{dt}|\psi\rangle = \dot{\hat{U}}(0)|\psi\rangle$

Now, from the unitary operator properties we have that $\dot{\hat{U}}^{\dagger}(0) = -\dot{\hat{U}}(0)$ that means that the derivative on time of a unitary operator is Antihermitian, while $\left(i\dot{\hat{U}}(0)\right)^{\dagger} = \dot{\hat{U}}^{\dagger}(0)i^{\dagger} = -\dot{\hat{U}}(0)\,(-i) = i\dot{\hat{U}}(0)$ that is Hermitian! (The i has the role to conserve probability!)
So, we can say that $i\dot{\hat{U}}(0)$ live in the class of Hermitian Operator, that we can call $\hat{H}$

Continuing from the equation we derived:

$\frac{d}{dt}|\psi\rangle = \dot{\hat{U}}(0)|\psi\rangle \rightarrow \frac{d}{dt}|\psi\rangle = \frac{\hat{H}}{i}|\psi\rangle \rightarrow i\frac{d}{dt}|\psi\rangle = \hat{H}\,|\psi\rangle$ and we found the **general form of the Schrodinger Equation**

Now, we know that a time variation corresponds to an energy operator, so $\hat{H}$, called Hamiltonian is somehow related to energy.
If we look close to this equation $i\frac{d}{dt}|\psi\rangle = \hat{E}\,|\psi\rangle$ it says $[s^{-1}] = J$, so we should make a correction of unit. This correction is indeed:

$$i\,\hbar\frac{d}{dt}|\psi\rangle = \hat{E}\,|\psi\rangle$$

Operators and Schrodinger Equations
https://www.youtube.com/watch?v=A7yDvA8VQC8&t=46s

We have found that the general Schrodinger Equation related to a quantum state, is written as:

$$\boldsymbol{i\,\hbar\frac{d}{dt}|\psi\rangle = \widehat{H}\,|\psi\rangle = \left(\frac{\hat{p}^2}{2m} + V(\hat{x})\right)\,|\psi\rangle}$$

Now we want to find three versions of it, specifically related to the position state, momentum state and energy state.

First, let's derive the definition of the operators we are interested in:
In a similar way to what we have done before, we start introducing the translation operator $\hat{T}(x_0)$. The translation operator works like: $\hat{T}(x_0)|x\rangle = |x + x_0\rangle$ and applied to a general quantum state: $\hat{T}(x_0)|x\rangle = \hat{T}(x_0)\int dx\,\psi(x)|x\rangle = \int dx\,\psi(x)\hat{T}(x_0)|x\rangle = \int dx\,\psi(x)|x + x_0\rangle$
This operator shifts all the position ket of a quantum state and intuitively we can say that this operator is reversable, and the probability is maintained, so that the translation operator is unitary. $\langle\hat{T}\psi|\hat{T}\psi\rangle = \langle\psi|\psi\rangle$

As we did with the time evolution operator, we look at the action of the translation operator over a small translation dx:
$\hat{T}(dx) = \hat{T}(0) + \dot{\hat{T}}(0)dx + \hat{O}(dx^2)$
$\hat{T}(dx)|x\rangle = \hat{I}|x\rangle + \dot{\hat{T}}(0)|x\rangle dx + \hat{O}(dx^2)|x\rangle \rightarrow (\dots) \rightarrow \frac{|x(dx))-|x\rangle}{dx} = \frac{\dot{\hat{T}}(0)dx|x\rangle + \hat{O}(dx^2)|x\rangle}{dx}$ and taking the limit of x:
$\frac{d}{dx}|x\rangle = \dot{T}(0)|x\rangle \rightarrow \; i\frac{d}{dx}|x\rangle = \hat{H}|x\rangle$

Now we have found once again the Hamiltonian. In this case, from classical physics, we derive that a variation of position corresponds somewhat to the momentum. In addition, we have to correct the unity by a factor $\hbar$ so that $[m^{-1}][\hbar = Js] = [Js\,m^{-1}]$ and we find how the momentum operator acts on a position ket:

$$i\,\hbar\frac{d}{dx}|x\rangle = \hat{p}\,|x\rangle$$

So, time evolution give us the Schrodinger equation with energy, space translation gives the action of the momentum operator, and it can be demonstrated that momentum translation gives the action of negative position operator and rotational changes give us the action of the angular momentum operator. In brief:

$$i\,\hbar\frac{d}{dt}|\psi\rangle = \widehat{H}\,|\psi\rangle \qquad i\,\hbar\frac{d}{dx}|x\rangle = \hat{p}\,|x\rangle \qquad i\,\hbar\frac{d}{dp}|p\rangle = -\hat{x}\,|p\rangle \qquad i\,\hbar\frac{d}{d\theta}|\theta\rangle = \hat{L}\,|\theta\rangle$$

Now, let's see how the momentum operator affects the position wavefunction:
$\hat{p}\,|\psi\rangle = \hat{p}\int dx\,\psi(x)|x\rangle = \int dx\,\psi(x)\hat{p}\,|x\rangle = \int dx\,\psi(x)\;\; i\hbar\frac{d}{dx}\,|x\rangle = \cdots = \int dx\left(-i\hbar\frac{d}{dx}\psi(x)\right)|x\rangle$, so $\langle x|\hat{p}|\psi\rangle = \; -i\hbar\frac{d}{dx}\,\psi(x)$

Knowing that we can rewrite the **Schrodinger equation in the position basis**:
$i\,\hbar\frac{d}{dt}|\psi\rangle = \widehat{H}\,|\psi\rangle = \left(\frac{\hat{p}^2}{2m} + V(\hat{x})\right)\,|\psi\rangle$
$i\,\hbar\frac{d}{dt}\langle x|\psi\rangle = \left(\frac{\hat{p}^2}{2m} + V(\hat{x})\right)\,\langle x|\psi\rangle$
$i\,\hbar\frac{\mathfrak{d}}{\mathfrak{d}t}\psi(x,t) = \left\langle x\left|\frac{\hat{p}^2}{2m}\right|\psi\right\rangle + \langle x|V(\hat{x})|\psi\rangle = \left(-i\hbar\frac{\mathfrak{d}}{\mathfrak{d}x}\right)^2\frac{1}{2m}\;\psi(x,t) + V(x)\,\psi(x,t)$, overall, we have:

$$i\,\hbar\frac{\mathfrak{d}}{\mathfrak{d}t}\psi(x,t) = \; -\frac{\hbar^2}{2m}\frac{\mathfrak{d}^2}{\mathfrak{d}x^2}\;\psi(x,t) + V(x)\,\psi(x,t)$$

We can also find the **Schrodinger equation in the momentum basis**:
$\langle p|\hat{x}|\psi\rangle = \; i\hbar\frac{d}{dp}\,\psi(p)$ can be derived as before.
$i\,\hbar\frac{d}{dt}\langle p|\psi\rangle = \left\langle p\left|\frac{\hat{p}^2}{2m}\right|\psi\right\rangle + \langle p|V(\hat{x})|\psi\rangle \rightarrow \; i\,\hbar\frac{\mathfrak{d}}{\mathfrak{d}t}\psi(p,t) = \frac{p^2}{2m}\,\psi(p,t) + V\left(i\hbar\frac{\mathfrak{d}}{\mathfrak{d}p}\right)\,\psi(p,t)$. Overall:

$$i\,\hbar\frac{\mathfrak{d}}{\mathfrak{d}t}\psi(p,t) = \frac{p^2}{2m}\,\psi(p,t) + V\left(i\hbar\frac{\mathfrak{d}}{\mathfrak{d}p}\right)\,\psi(p,t)$$

Finally we find the **Schrodinger equation in the energy basis**:
$i\,\hbar\frac{d}{dt}\langle E_i|\psi\rangle = \langle E_i|\hat{H}|\psi\rangle \;\rightarrow \;\; i\,\hbar\frac{d}{dt}\,c_i(t) = E_i\,c_i(t)$ assuming that the Hamiltonian is time-independent

$$i\,\hbar\frac{d}{dt}\,c_i(t) = E_i\,c_i(t)$$

Alle these different representations represent the same equation and describes how the coefficients in a particular basis evolve in time.

2) Position and Momentum Representation

https://www.youtube.com/watch?v=ApHzbZja0sI

2.1) Position and Momentum: Classical Physics vs Quantum Physics and interpretation of a waveform

In **classical physics**, the position and momentum of a particle are well-defined quantities at any given time. It is possible to know exactly where a particle is (its spatial coordinates) and how fast it is moving (its momentum $p = mv$) without any uncertainty. The trajectory of an object over time is completely determined by its initial conditions.

In **quantum physics**, however, position and momentum are not simultaneously well-defined (we saw it from the concept of the commutator).
When Schrodinger derived his equation, he thought that a quantum particle disintegrates and the wavefunction $\psi(x)$ indicates the concentration of the disintegrated particle.
Then came Bohr and he affirmed that "$\psi(x,t)$ does not tell how much of the particle is at x at time t but rather the probability to find it at x at time t".
Bohr's interpretation turned out to be correct, even though Schrödinger and Einstein never fully accepted the statistical approach (Schrödinger even devised his famous cat paradox to highlight what he saw as the absurdity of quantum superposition at macroscopic scales). In fact:

$$P(x) = |\psi(x)|^2 \qquad P(p) = \left|\tilde{\psi}(p)\right|^2$$

This probabilistic view marks a fundamental shift from classical physics: we can no longer speak of position and momentum as certain numbers, but only of probabilities associated with possible measurement outcomes.

2.2) Statistical Concepts: variance and mean value

Since quantum measurements are inherently probabilistic, we cannot associate a fixed value to an observable like position or momentum. Instead, what we obtain from a measurement is a distribution of possible outcomes.
To make sense of this distribution, we rely on two important statistical concepts: the mean value (or expectation value) and the variance.
The **mean value** represents the average result we would obtain by repeating the same measurement on many identical systems. For the position of a particle, the mean value is given by:

$$\langle x \rangle = \langle \psi | \hat{x} | \psi \rangle = \int_{-\infty}^{+\infty} x\, |\psi(x)|^2\, dx\, x = \int_{-\infty}^{+\infty} x\, P(x)\, dx$$

However, the mean value alone does not tell us how "spread out" the possible outcomes are. This is why we introduce the variance, which measures how much the individual results typically deviate from the mean value.
For position, the **variance** is defined as:

$$(\Delta x)^2 = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$$

NB: In classical physics we indicate the mean value and the variance as $\bar{x}$ and σ^2, while in quantum physics as $\langle x \rangle$ and $(\Delta x)^2$

2.3) From $\psi(x)$ to $\tilde{\psi}(p)$: Fourier Transform in space coordinates

Now we want to find how to pass from the wavefunction in space domain to wavefunction in momentum domain.
We are interested in two main reasons: we want to exploit information from both the domains and sometimes it's easier to work on a domain respect to the other.
Starting from the space Fourier transform, we have:

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int \Phi(k) e^{ikx} dk \qquad \Phi(k) = \frac{1}{\sqrt{2\pi}} \int \Psi(x) e^{-ikx} dx$$

And we must underline that $\Phi(k)$ carries the same information of $\Psi(x)$ and $\Phi(k)$ gives the weight of each plain wave that compose $\Psi(x)$ and vice versa.
But we were interested in the momentum, not in the wave vector. Knowing that $p = \hbar k$ we have:
$\Phi\left(\frac{p}{\hbar}\right) = \frac{1}{\sqrt{2\pi}} \int \Psi(x) e^{-\frac{ip}{\hbar}x} dx$ now, since we want it to be normalized, the correct expressions are:

$$\boldsymbol{\Psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int \Phi(p) e^{\frac{ip}{\hbar}x} dp} \qquad \boldsymbol{\tilde{\Psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \Psi(x) e^{-\frac{ip}{\hbar}x} dx}$$

Now we want to prove the same thing through a different way, a more mathematical approach.
We start from a define momentum state that can be written as: $\hat{p}|p\rangle = p_0|p\rangle$, so the operator momentum, a wavefunction, applied to the ket momentum, gives an eigenvalue p_0 still multiplied by the ket momentum.

Now we want to see the same in position representation, so we apply the bra position to both terms:

$\langle x\|\hat{p}\|p\rangle = \langle x\|p_0\|p\rangle$	Now we can move out the operator and the constant value
$\hat{p}\ \langle x\|p\rangle = p_0\langle x\|p\rangle$	We call $\langle x\|p\rangle$ as $U_p(x)$
$\hat{p}\ U_p(x) = p_0\ U_p(x)$	From the math of quantum mechanics, we know that $\hat{p}\ U_p(x) = i\hbar\frac{dU_p(x)}{dx}$
$i\hbar\frac{dU_p(x)}{dx} = p_0\ U_p(x)$	And we have to solve this differential equation. From calculus we say $U_p(x) = A\ e^{i\frac{p_0}{\hbar}x}$

Now we aim to determine the constant A in order to properly normalize the wave function. It is important to highlight that, in this case, we cannot simply set the normalization integral equal to 1, as it is not possible to normalize eigenstates of continuous physical quantities to unity in quantum mechanics.
Unlike discrete systems, where normalizable states have a finite norm, here the states form a continuous spectrum and are not square-integrable. For this reason, we impose that the normalization condition is equal to a Dirac delta function, reflecting the orthogonality and completeness properties typical of continuous bases.

$\int_{-\infty}^{+\infty} dx\ U_p^*(x)U_p(x) = \delta(p-p')$

$|A|^2\hbar\ \int_{-\infty}^{+\infty} dx\frac{1}{\hbar}\ e^{i(p-p')\frac{x}{\hbar}} = \delta(p-p')$ that integral is a complex exponential that oscillates and gives:

$|A|^2\hbar\ 2\pi\ \delta(p-p') = \delta(p-p')$ at the end of the day we have found that $A = \frac{1}{\sqrt{2\pi\hbar}}$, resulting:

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\ e^{i\frac{p}{\hbar}x} \qquad \langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}}\ e^{-i\frac{p}{\hbar}x}$$

Now, given a quantum state ψ, the wavefunction in position representation is $\tilde{\psi}(p) = \langle p|\psi\rangle$, so the quantum state can be written as:

$$|\psi\rangle = \int_{-\infty}^{+\infty} dp\ \langle p|\psi\rangle\ |p\rangle = \int_{-\infty}^{+\infty} dp\ \tilde{\phi}(p)\ |p\rangle$$

Now we want to find its position representation:

$$\langle x|\psi\rangle = \int_{-\infty}^{+\infty} dp\ \tilde{\phi}(p)\ \langle x|p\rangle = \int_{-\infty}^{+\infty} dp\ \tilde{\phi}(p)\ \frac{1}{\sqrt{2\pi\hbar}}\ e^{i\frac{p}{\hbar}x}$$

And so, we have arrived at the same expression:

$$\boldsymbol{\psi(x) = \langle x|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} dp\ \tilde{\phi}(p)\ e^{i\frac{p}{\hbar}x}} \qquad \boldsymbol{\tilde{\psi}(p) = \langle p|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} dx\ \psi(x)\ e^{-i\frac{p}{\hbar}x}}$$

2.4) From P(x) to P(p) : The Heisenberg Uncertainty Principle

In quantum mechanics, the description of a system is fully encoded in its wavefunction, but often we are particularly interested in the probability densities associated with different measurements. As we have seen $P(x) = |\psi(x)|^2$.
Since the two wavefunctions are connected through Fourier Transform, we can assume that also the two probabilities are connected.

We start considering $P(x)$ as a gaussian shape since it's mathematically convenient and physically reasonable.

$$P(x) = \frac{e^{-\frac{x^2}{2(\Delta x)^2}}}{\sqrt{2\pi(\Delta x)^2}}$$

From this, we know derive the wavefunction in position representation:

$$\psi(x) = \sqrt{P(x)} = \frac{e^{-\frac{x^2}{4(\Delta x)^2}}}{(2\pi(\Delta x)^2)^{\frac{1}{4}}}$$

Now through Fourier Transform we move to the wavefunction in momentum representation:

$$\tilde{\psi}(p) = \int_{-\infty}^{+\infty} dx\ \psi(x)\ \frac{1}{\sqrt{2\pi\hbar}}\ e^{-i\frac{p}{\hbar}x} = \int_{-\infty}^{+\infty} dx\ \frac{e^{-\frac{x^2}{4(\Delta x)^2}}}{(2\pi(\Delta x)^2)^{\frac{1}{4}}}\ \frac{1}{\sqrt{2\pi\hbar}}\ e^{-i\frac{p}{\hbar}x} = \frac{e^{-\frac{(\Delta x)^2p^2}{\hbar^2}}}{\left(\frac{2\pi\hbar^2}{4(\Delta x)^2}\right)^{\frac{1}{4}}}$$

And finally

$$P(p) = \left(\tilde{\psi}(p)\right)^2 = \frac{e^{-\frac{2(\Delta x)^2p^2}{\hbar^2}}}{\left(\frac{2\pi\hbar^2}{4(\Delta x)^2}\right)^{\frac{1}{2}}} = \left[\Delta p = \frac{\hbar}{2\Delta x}\right] = \frac{e^{-\frac{p^2}{2(\Delta p)^2}}}{\sqrt{2\pi(\Delta p)^2}}$$

And we find the Heisenberg Uncertainty Principle:

$$\boldsymbol{\Delta p\Delta x = \frac{\hbar}{2}}$$

In grey the resolution of the monstrous integral:

$$\tilde{\psi}(p) = \int_{-\infty}^{+\infty} dx \; \frac{e^{-\frac{x^2}{4(\Delta x)^2}}}{(2\pi(\Delta x)^2)^{\frac{1}{4}}} \frac{1}{\sqrt{2\pi\hbar}} \; e^{-i\frac{p}{\hbar}x} = \frac{1}{\sqrt{2\pi\hbar}\,(2\pi(\Delta x)^2)^{\frac{1}{4}}} \int_{-\infty}^{+\infty} dx \; e^{-\left(\frac{x^2}{4(\Delta x)^2} + i\frac{p}{\hbar}x - \frac{p^2}{\hbar^2}(\Delta x)^2\right)} e^{-\frac{p^2}{\hbar^2}(\Delta x)^2} = \frac{e^{-\frac{p^2}{\hbar^2}(\Delta x)^2}}{\sqrt{2\pi\hbar}\,(2\pi(\Delta x)^2)^{\frac{1}{4}}} \int_{-\infty}^{+\infty} dx \; e^{-\left(\frac{x}{2\Delta x} + i\frac{p}{\hbar}\Delta x\right)^2}$$

Given that $\int_{-\infty}^{+\infty} e^{-f^2(x)}\, dx = \frac{1}{f'(x)}\sqrt{\pi}$ $\quad\rightarrow\quad$ $\tilde{\psi}(p) = \frac{e^{-\frac{p^2}{\hbar^2}(\Delta x)^2}\; 2\Delta x\sqrt{\pi}}{\sqrt{2\pi\hbar}\,(2\pi(\Delta x)^2)^{\frac{1}{4}}} = \frac{e^{-\frac{(\Delta x)^2 p^2}{\hbar^2}}}{\left(\frac{2\pi\hbar^2}{4(\Delta x)^2}\right)^{\frac{1}{4}}}$

2.5) Time evolution of Position and Momentum

In quantum mechanics, the time evolution of position and momentum reveals how a system behaves over time. Unlike classical mechanics, where particles follow predictable paths, quantum particles are described by evolving wavefunctions. By studying how the expectation values of position $\langle x(t)\rangle$ and momentum $\langle p(t)\rangle$ evolve, we can uncover the quantum dynamics of a system.

We first derive how the position operator changes over time:

$$\frac{d}{dt}\hat{p} = \frac{d}{dt}\langle\psi|\hat{p}|\psi\rangle = \left\langle\frac{d}{dt}\psi|\hat{p}|\psi\right\rangle + \left\langle\psi|\hat{p}|\frac{d}{dt}\psi\right\rangle$$

From the quantum math we know that: $i\hbar\frac{d}{dt}\,|\psi\rangle = \hat{H}\,|\psi\rangle$

$$\frac{d}{dt}\hat{p} = -\frac{1}{i\hbar}\,\langle\psi|\hat{H}\hat{p}|\psi\rangle + \frac{1}{i\hbar}\langle\psi|\hat{p}\hat{H}|\psi\rangle = \frac{\langle\psi|\hat{p}\hat{H}|\psi\rangle - \langle\psi|\hat{H}\hat{p}|\psi\rangle}{i\hbar} = \frac{\langle\psi|[\hat{p},\hat{H}]|\psi\rangle}{i\hbar}$$

We exploit the commutator between the momentum operator and the Hamiltonian

$$[\hat{p},\hat{H}] = \left[\hat{p},\frac{\hat{p}^2}{2m} + \hat{V}(x)\right] = \left[\hat{p},\frac{\hat{p}^2}{2m}\right] + [\hat{p},\hat{V}(x)] = 0 + [\hat{p},\hat{V}(x)]$$

So, we understand that we have to calculate the commutator between the momentum and the potential energy. We can use Taylor:

$$\hat{V}(x) = V(\hat{x}) = V_0 + V_1(\hat{x}) + \tfrac{V_2}{2}\,(\hat{x}^2) + \cdots$$

$$[\hat{p},\hat{V}(x)] = [\hat{p},V_0] + [\hat{p},V_1(\hat{x})] + \left[\hat{p},\frac{V_2}{2}\hat{x}^2\right] + \cdots = 0 + V_1[\hat{p},\hat{x}] + \frac{V_2}{2}\{[\hat{p},\hat{x}]\hat{x} + \hat{x}[\hat{p},\hat{x}]\} + \cdots = [\hat{p},\hat{x}]\left\{V_1 + V_2\hat{x} + \frac{V_3}{2}\hat{x}^2 + \cdots\right\} = [\hat{p},\hat{x}]\frac{dV}{dx}$$

Since from quantum math we have that the operator p is defined as $\hat{p} = -i\hbar\frac{d}{dx}$

$$[\hat{p},\hat{x}] = \hat{p}\hat{x} - \hat{x}\hat{p} = -i\,\hbar\frac{d}{dx}\,\hat{x} + \hat{x}i\,\hbar\frac{d}{dx} = -i\hbar + 0 = -i\hbar$$

So, in the end we have:

$$\frac{d}{dt}\hat{p} = \frac{\langle\psi|[\hat{p},\hat{H}]|\psi\rangle}{i\hbar} = \frac{\langle\psi\left|[\hat{p},\hat{x}]\frac{dV}{dx}\right|\psi\rangle}{i\hbar} = \langle\psi\left|\frac{dV}{dx}\right|\psi\rangle$$

In a similar way it can be proved that:

$$\frac{d}{dt}\hat{x} = \frac{\langle\psi|[\hat{x},\hat{H}]|\psi\rangle}{i\hbar} = \frac{\langle\psi\left|\frac{i\hbar}{m}\,\hat{p}\right|\psi\rangle}{i\hbar} = \frac{\langle\psi|\hat{p}|\psi\rangle}{m}$$

With

$$[\hat{x},\hat{H}] = \left[\hat{x},\frac{\hat{p}^2}{2m}\right] + [\hat{x},V(\hat{x})] = \left[\hat{x},\frac{\hat{p}^2}{2m}\right] + 0 \quad \text{and} \quad \frac{[\hat{x},\hat{p}^2]}{2m} = \frac{1}{2m}\{[\hat{x},\hat{p}]\hat{p} + \hat{p}[\hat{x},\hat{p}]\} = \frac{2i\hbar}{2m}\,\hat{p}$$

2a) Ex07: Well Defined Position ----------

Consider an electron in a quantum state with well-defined position (at a given instant time) in a point P described by the position vector $r_p = (x_p, y_p, z_p)$.

a) Write the general expression of a wave function in position representation for a quantum state describing an electron with a well-defined position

b) Verify that the above well-defined position quantum state is an eigenstate of the position vector operator and calculate the corresponding eigenvalues

c) Calculate the expectation value (mean value) and the uncertainty of the position and momentum observable in case of the above quantum state

d) Write the quantum state in position representation

a) Write the general expression of a wave function in position representation for a quantum state describing an electron with a well-defined position

A quantum state with well-defined position in position representation (wavefunction) can be written as:

$\psi(\bar{r}) = \delta(\bar{r} - \bar{r_p}) \quad with\ \bar{r_p} = (x_p, y_p, z_p), so\ \ \psi(x,y,z) = \delta(x - x_p)\,\delta(y - y_p)\,\delta(z - z_p)$

b) Verify that the above well-defined position quantum state is an eigenstate of the position vector operator and calculate the corresponding eigenvalues

In fact, for a well-defined position, we only have one eigenstate that corresponds at that value. All $f(x)$ multiplied by $\delta(\bar{r} - \bar{r_p})$ will make the position collapse in $\bar{r_p}$, so $\psi(\bar{r}) = \delta(\bar{r} - \bar{r_p})$ is eigenfunction of $\hat{x}, \hat{y}, \hat{z}$ with eigenvalues x_p, y_p, z_p

$\hat{x}|\psi(\bar{r})\rangle = x_p|\psi(\bar{r})\rangle$

$\hat{y}|\psi(\bar{r})\rangle = y_p|\psi(\bar{r})\rangle$

$\hat{z}|\psi(\bar{r})\rangle = z_p|\psi(\bar{r})\rangle$

c) Calculate the expectation value (mean value) and the uncertainty of the position and momentum observable in case of the above quantum state

The mean value is defined as:

$$\langle x\rangle = \frac{\int_{-\infty}^{+\infty} dxdydz\ x\,|\psi(x,y,z)|^2}{\int_{-\infty}^{+\infty} dxdydz\ |\psi(x,y,z)|^2} \qquad \langle y\rangle = \frac{\int_{-\infty}^{+\infty} dxdydz\ y\,|\psi(x,y,z)|^2}{\int_{-\infty}^{+\infty} dxdydz\ |\psi(x,y,z)|^2} \qquad \langle z\rangle = \frac{\int_{-\infty}^{+\infty} dxdydz\ z\,|\psi(x,y,z)|^2}{\int_{-\infty}^{+\infty} dxdydz\ |\psi(x,y,z)|^2}$$

That in our case are:

$$\langle x\rangle = \frac{\int_{-\infty}^{+\infty} dxdydz\ x\,|\psi(x,y,z)|^2}{\int_{-\infty}^{+\infty} dxdydz\ |\psi(x,y,z)|^2} = \frac{\int_{-\infty}^{+\infty} dxdydz\ x\,\delta^2(x-x_p)\delta^2(y-y_p)\delta^2(z-z_p)}{\int_{-\infty}^{+\infty} dxdydz\ \delta^2(x-x_p)\delta^2(y-y_p)\delta^2(z-z_p)} = \frac{x_p\int_{-\infty}^{+\infty} dxdydz\ \delta^2(x-x_p)\delta^2(y-y_p)\delta^2(z-z_p)}{\int_{-\infty}^{+\infty} dxdydz\ \delta^2(x-x_p)\delta^2(y-y_p)\delta^2(z-z_p)} = x_p$$

In the same way for y and z:

$$\langle y\rangle = \frac{\int_{-\infty}^{+\infty} dxdydz\ y\,|\psi(x,y,z)|^2}{\int_{-\infty}^{+\infty} dxdydz\ |\psi(x,y,z)|^2} = y_p$$

$$\langle z\rangle = \frac{\int_{-\infty}^{+\infty} dxdydz\ z\,|\psi(x,y,z)|^2}{\int_{-\infty}^{+\infty} dxdydz\ |\psi(x,y,z)|^2} = z_p$$

NB: For a quantum state well defined, we restore the classical physics results!

Now, the uncertainty or variance is defined as:

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} \qquad \Delta y = \sqrt{\langle y^2\rangle - \langle y\rangle^2} \qquad \Delta z = \sqrt{\langle z^2\rangle - \langle z\rangle^2}$$

That in our cases are:

$$\langle x^2\rangle = \frac{\int_{-\infty}^{+\infty} dxdydz\ x^2\,|\psi(x,y,z)|^2}{\int_{-\infty}^{+\infty} dxdydz\ |\psi(x,y,z)|^2} = x_p^2$$

$\langle x\rangle = x_p \ \rightarrow \ \ \langle x\rangle^2 = x_p^2$

For all the coordinates resulting:

$$\Delta x = x_p^2 - x_p^2 = 0 \qquad \Delta y = y_p^2 - y_p^2 = 0 \qquad \Delta z = z_p^2 - z_p^2 = 0$$

From the Heisenberg Uncertanty Principle we know that $\Delta x \Delta p_x \geq \frac{\hbar}{2}$, so we find:

$\Delta p_x \rightarrow \infty$

$\Delta p_y \rightarrow \infty$

$\Delta p_z \rightarrow \infty$

d) Write the quantum state in position representation

From the theory we know that the wavefunction in position representation and the wavefunction in momentum representation are linked with a special Fourier transform

$$\tilde{\psi}(p) = \langle p|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx\ \psi(x)\ e^{-i\frac{p}{\hbar}x}$$

So, in our case we have:

$\tilde{\psi}(p_x, p_y, p_z) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx\ \delta(x - x_p)\ e^{-i\frac{p}{\hbar}x}\ \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dy\ \delta(y - y_p)\, e^{-i\frac{p}{\hbar}y} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dz\ \delta(z - z_p)\ e^{-i\frac{p}{\hbar}z} = \frac{1}{\sqrt{\hbar^3}}\ e^{-\frac{i}{\hbar}(p_x x_p + p_y y_p + p_z z_p)}$

This expression of the momentum waveform corresponds to a plane wave! In fact the probability is spread over all the momentum space and the uncertainty tends to infinity.

2a) Ex08: Well Defined Momentum ----------

Consider a quantum state for an electron with well-defined momentum corresponding to a well-defined velocity $v_z = 3 \cdot 10^6\ m/s$.

a) Write the wave function for both momentum and position representation
b) Verify that the above quantum state is an eigenstate of the momentum vector operator and determinate the corresponding eigenvalues of the three components of momentum operator
c) Determine the expectation value (mean value) and uncertainty of both momentum and position

a) Write the wave function for both momentum and position representation

We understood that in this case we should have a delta in the momentum representation and a plane wave in the position representation. Let's start with the plane wave that should be:

$\psi(x, y, z) \propto e^{\frac{i}{\hbar}(p'_x x + p'_y y + p'_z z)} \propto e^{\frac{i}{\hbar}(p'_z z)}$ since the particle is moving along the z direction

To pass to the momentum representation we do the Fourier Transform:

$$\tilde{\psi}(p_x, p_y, p_z) \propto \int_{-\infty}^{+\infty} dxdydz\ e^{\frac{i}{\hbar}(p'_z z)} e^{-i\frac{(p_x x + p_y y + p_z z)}{\hbar}} = \int dx\, e^{-\frac{i}{\hbar}p_x x} \int dy\, e^{-\frac{i}{\hbar}p_y y} \int dz\, e^{-\frac{i}{\hbar}(p_z - p'_z)z} \propto \delta(p_x)\delta(p_y)\delta(p_z - p'_z) = \delta(\bar{p} - \overline{p_0})$$

b) Verify that the above quantum state is an eigenstate of the momentum vector operator and determinate the corresponding eigenvalues of the three components of momentum operator

For the above quantum state, since the multiplication for a delta makes the state collapse to the value, we find:

$\widehat{p_x}|\tilde{\psi}(\bar{p})\rangle = p_x \delta(p_x)\delta(p_y)\delta(p_z - p'_z) = 0$ because $\delta(p_x)$ is non null for $p_x = 0$
$\widehat{p_y}|\tilde{\psi}(\bar{p})\rangle = p_y \delta(p_x)\delta(p_y)\delta(p_z - p'_z) = 0$ same as before
$\widehat{p_z}|\tilde{\psi}(\bar{p})\rangle = p_z \delta(p_x)\delta(p_y)\delta(p_z - p'_z) = p'_z|\tilde{\psi}(\bar{p})\rangle = m\, v_z|\tilde{\psi}(\bar{p})\rangle$ because $\delta(p_z - p'_z)$ is non null for $p_z = p'_z$

Otherwise, we can use position representation to verify: (we should remember that $\widehat{p_x} = -i\hbar\frac{d}{dx}$, etc.)

$\widehat{p_x}|\psi(\bar{r})\rangle = -i\hbar\frac{d}{dx}\left(e^{\frac{i}{\hbar}(p'_x x + p'_y y + p'_z z)}\right) \propto p'_x\ |\psi(\bar{r})\rangle = 0$
$\widehat{p_y}|\psi(\bar{r})\rangle = -i\hbar\frac{d}{dy}\left(e^{\frac{i}{\hbar}(p'_x x + p'_y y + p'_z z)}\right) \propto p'_y\ |\psi(\bar{r})\rangle = 0$
$\widehat{p_z}|\psi(\bar{r})\rangle = -i\hbar\frac{d}{dz}\left(e^{\frac{i}{\hbar}(p'_x x + p'_y y + p'_z z)}\right) \propto p'_z\ |\psi(\bar{r})\rangle = m\, v_z\ |\psi(\bar{r})\rangle$

NB: We can use any representation to verify properties.

c) Determine the expectation value (mean value) and uncertainty of both momentum and position

$\langle p_x\rangle = p'_x$ $\quad\quad$ $\Delta x \to \infty$
$\langle p_y\rangle = p'_y$ $\quad\quad$ $\Delta y \to \infty$
$\langle p_z\rangle = p'_z$ $\quad\quad$ $\Delta z \to \infty$

QUBIT

1. **All you need to know about Qubit**
2. **Block Sphere and Pauli Operators**
3. **Fundamentals for Quantum Cryptography**
4. **Quantum Key Distribution**

Cool Materials you may be interested in:

3Blue1Brown (Quantum Computing):
https://www.youtube.com/watch?v=RQWpF2Gb-gU&t=1760s
https://www.youtube.com/watch?v=Dlsa9EBKDGI

Looking Glass Universe (Quantum Mechanics):
What is a Qubit? : https://www.youtube.com/watch?v=kgSVkVNxXyU

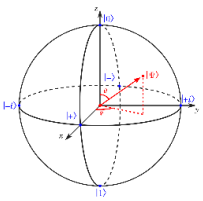
Appunti di Sara
Una rivisitazione del corso di Quantum Communication

THESE ARE **NOT** THE NOTES OF THE QUANTUM COMMUNICATION COURSE.
SEE THE LAST PAGE FOR MORE DETAILS.

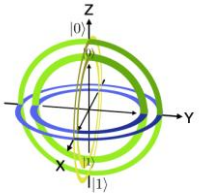
1. All you need to know about Qubit

Check all the website: **"Quantum computing for the very curious"** https://quantum.country/qcvc

2. Block Sphere and Pauli Operators



A quantum state—more formally a ray in a two-dimensional Hilbert space—can be represented as a point on the Bloch sphere, where its position is specified by the colatitude (θ) and the longitude (ϕ). The Bloch sphere is analogous to the unit circle in trigonometry, which serves as the geometrical representation of angles; here, it elegantly depicts the state space of a two-level quantum system.
The sphere is structured around three orthogonal axes, each corresponding to a different measurement basis. The intersections of these axes with the sphere yield six significant points: the North Pole, associated with the state |0⟩ (often linked to horizontal polarization in the context of light), while the South Pole represents |1⟩; the West Pole |+⟩ and the East Pole |-⟩; the Left Pole |i⟩ and the Right Pole |-i⟩.
In this framework, any two orthogonal qubits are represented by points that lie diametrically opposite each other on the sphere. Moreover, quantum operations — especially rotations generated by the Pauli operators — allow us to traverse the sphere, thereby transforming one quantum state into another. This geometric representation not only deepens our understanding of quantum superposition and coherence but also provides a powerful tool for visualizing and manipulating qubit states in quantum computation and information theory.



Click here for the GIF

A quantum gate $\hat{U}$ with orthogonal eigenstates can be characterized by two diametrically opposite points, (P_0) and (P_1), on the Bloch sphere. These points correspond to the eigenvalues $(\lambda_0 = e^{i\phi_0})$ and $(\lambda_1 = e^{i\phi_1})$. The effect of $(\hat{U})$ on the Bloch sphere is a rotation around the axis passing through P_0 and P_1, with an angle $\phi = \phi_0 - \phi_1$, which represents the phase difference between the eigenvalues.
We will study Pauli operators, including:

- **Z** (quantum NOT gate), which flips the phase of the qubit,
  1) **X**, which acts as a bit-flip operation,
- **Y**, which combines both bit-flip and phase-flip effects,
- **R** (phase shift gate), which modifies the phase of the quantum state,
- **H** (Hadamard gate), which creates equal superpositions of basis states, enabling quantum interference.

Each of these operators plays a crucial role in quantum computation, influencing how qubits evolve and interact within quantum circuits.

Z – $|0\rangle$ & $|1\rangle$ - N-S base

$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ N-S base

X – $|+\rangle$ & $|-\rangle$ - E-W base

$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ N-S base $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ E-W base

Y – $|i\rangle$ & $|-i\rangle$ - L-R base

$\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$ N-S base $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ R-L base

R (Z extended) – $|+\rangle$ & $|-\rangle$ - N-S base

$\begin{bmatrix}1 & 0\\0 & e^{i\theta}\end{bmatrix}$ N-S base

H – From Z to X

$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$ N-S base

3. Fundamentals for Quantum Cryptography

Quantum cryptography is built upon fundamental principles of quantum mechanics that ensure the security of key distribution. Before diving into Quantum Key Distribution (QKD), it is essential to understand these core concepts:

Compatible Observables
In quantum mechanics, observables are physical quantities that can be measured. Two observables are considered compatible if their measurements do not disturb each other.
Mathematically, this means that their corresponding operators commute: $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = 0$. For instance, position and momentum do not commute, leading to Heisenberg's Uncertainty Principle, whereas certain spin components might commute under specific conditions.
<u>The core concept is that two compatible observables are simultaneously measurable while two NOT compatible observables are NOT simultaneously measurable.</u>

Mutually Unbiased Bases (MUBs)
Mutually Unbiased Bases (MUBs) play a crucial role in quantum cryptography. Two bases are mutually unbiased if measuring a quantum state prepared in one basis in the other basis yields completely random outcomes. A well-known example involves the Pauli X and Pauli Z bases in qubit representation: If a qubit is prepared in the Z basis and measured in the X basis, the outcome is completely random (50% chance of getting $|+\rangle$ or $|-\rangle$). This randomness is fundamental to the security of QKD.

Complementary Observables

Observables are complementary when knowing the exact measurement outcome of one makes it impossible to predict the outcome of the other with certainty, respectively, with a probability of $\frac{1}{D}$ with D the dimension of the Hilbert Space.

Two Observables are complementary when they are compatible and the two basis formed by the respective eigenstates must be mutually unbiased. The three Pauli Operators X, Y, Z form a set of mutually complementary unbiased bases since they are complementary and their axis on the Bloch Spere are orthogonal.

4. Quantum Key Distribution

Quantum Key Distribution (QKD) is a groundbreaking method for secure communication that leverages the principles of quantum mechanics to exchange cryptographic keys. The **BB84 protocol**, introduced by Charles Bennett and Gilles Brassard in 1984, is one of the most widely used QKD schemes. It ensures security through the quantum properties of photons, particularly the concept of **mutually unbiased bases**, which prevent undetected eavesdropping.

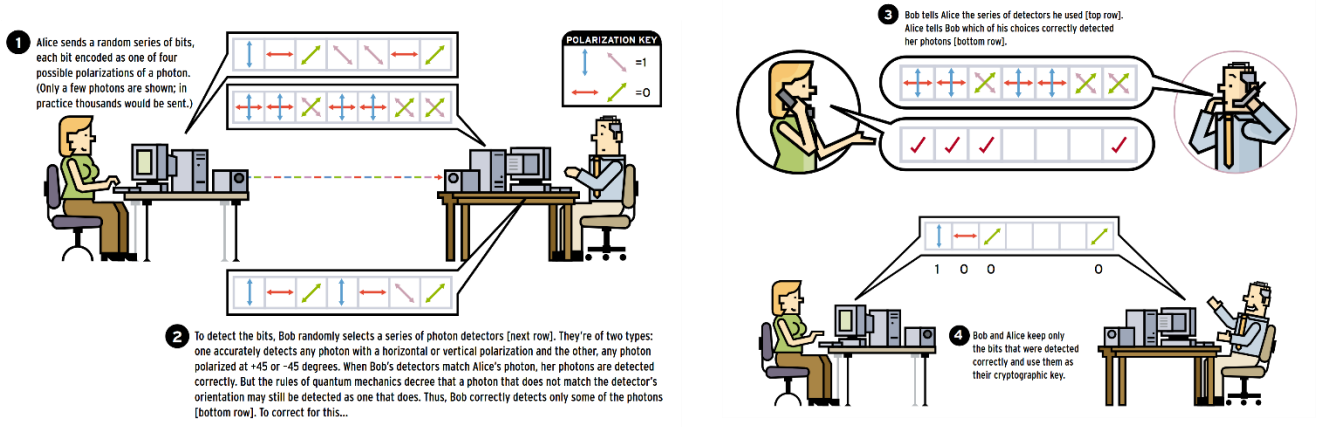
In BB84, **Alice** transmits quantum states encoded in one of two bases: the **rectilinear basis** (|0⟩) & (|1⟩) (horizontal and vertical polarization) and the **diagonal basis** (|+⟩) & (|-⟩) (diagonal polarization).

Bob, upon receiving these photons, randomly selects a basis for measurement. After transmission, Bob publicly informs Alice of the bases he used, and Alice reveals which measurements were made in the correct basis. Bob keeps only those bits that match, forming a shared key.

If an **eavesdropper (Eve)** attempts to intercept the photons, quantum mechanics ensures that her intrusion introduces errors due to the **no-cloning theorem** and the **collapse of quantum states upon measurement**. By analyzing the **Quantum Bit Error Rate (QBER),** typically around **25%** when eavesdropping occurs, Alice and Bob can assess the security of their key. To verify its integrity, they occasionally sacrifice a portion of their decrypted bits to compare the sent and received values. If discrepancies arise, it indicates that photons were absorbed and re-emitted in a **random basis**, potentially differing from Alice's original encoding, revealing Eve's presence.

BB84 provides a **provably secure** method for key distribution, making it a fundamental protocol in quantum cryptography.

From IEEE SPECTRUM • May 2002



CONCLUSIONS

The exploration of quantum communication begins with the foundations of quantum mechanics: the wave-particle duality, uncertainty principle, and the mathematical framework that underpins quantum theory. Through the formalism of Hilbert spaces, inner products, and operators, the theoretical structure necessary for understanding qubits emerges.
Qubits, unlike classical bits, introduce new possibilities through superposition and entanglement. The Bloch sphere provides a geometric representation of quantum states, while Pauli operators define key transformations in quantum systems. These principles form the basis for quantum computation and, crucially, quantum cryptography.
Quantum Key Distribution (QKD) leverages these unique quantum properties to ensure secure communication. The BB84 protocol demonstrates how mutually unbiased bases allow for the detection of eavesdroppers, providing a provably secure method for key exchange. The interplay between measurement, information disturbance, and classical verification techniques confirms that quantum communication is fundamentally different from classical encryption methods.
This journey through quantum mechanics, qubits, and cryptographic protocols establishes a framework for further advancements. Quantum networks, quantum-secure communication, and future innovations will continue to shape the field, bridging theoretical developments with practical applications. The fundamental principles outlined here serve as a foundation for deeper study and technological breakthroughs in quantum information science.

PS: These are not the notes from the Quantum Communication course (Polimi, 2024/2025). Instead, starting from the notes of this course (hoping to have covered everything), I have enriched them with additional mathematical formalism, as I believed it was the right approach to truly understand the subject. While not all of it is strictly necessary, I consider it useful for a student trying to determine their level of interest in the topic and whether it could be part of their future studies or career.
I do not guarantee the absolute accuracy of these notes, but I tried to put everything together in a coherent and well-structured resource. Hopefully this will be useful to someone else <3.

Appunti di Sara :) e un pizzico di Copilot
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