Formulary of Optical Measurement

Tommaso Dezuanni

January 10, 2022

Contents

1 Laser Sources and Principle of Operation

1.1 Absorption, spontaneous emission, stimulated emission

Do consider an ideal atom characterized by two possibile energy levels, known as 2-level model. When a photon emitted by the atom, the electrons involved in this dynnamic correspond to this energy transition.

$$U_{upper} - L_{low} = E_{ph} \Rightarrow E_2 - E_1 = h\nu$$

When the photon is absorbed the electron get excited and transits from the lower level to the upper one. Viceversa, if the electron decays, it releases energy in the form of photons.

The stimulated emission is the particular mechanism that happens inside an amplifier: the atomic concentration of the upper energy level is higher than that of the lower one, $N_2 > N_1$. When this condition, called *population inversion*, is satisfied, the incident photon causes a *coherent* emission: the emitted photon has the same frequency and phase of the incident one. Besides, if $N_2 = N_1$ the material is said to be "transparent".

1.2 Laser Oscillation

We need an active medium, a pump mechanism and the physical structure to allow the electromagnetic radiation to oscillate: it's called (*cavity*). Inside the cavity the losses are caused by diffraction and absorpton, when gain is equal to total losses the oscillator is said to be stable (by definition).

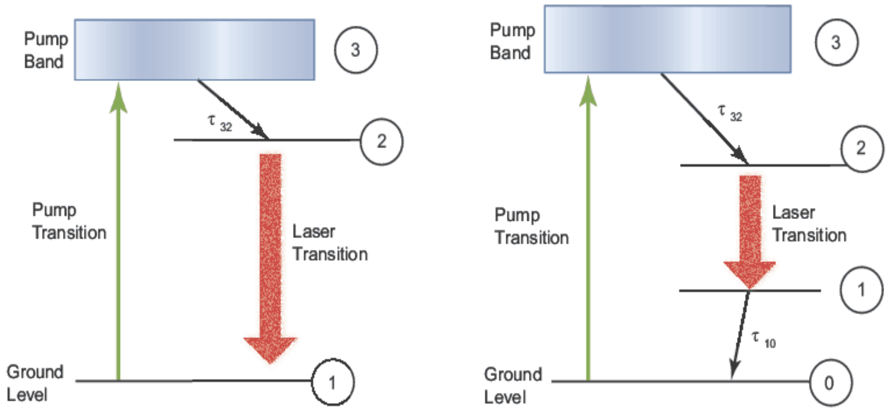
Optical length: $L_{opt} = (L - l)1 + l\, n_{am} = L + l(n_{am} - 1)$

Round trip: $\tau_{rt} = \frac{2L_{opt}}{c}$

1.3 Optical Amplification

First of all, with the 2-levels model, in the case of $N_1 = N_2 = {}^{N_0}/_2$, the stimulated emission is not achievable because the probabilties of emission and absorption are equal. Furthermore we cannot pump N_1 higher than ${}^{N}/_2$, so we need more levels

For example, the ruby is used to obtain the classic red laser that we all know, and it's a 3-level model. It emits light at $\lambda = 690nm$. In this case the active medium is realized doping the Al_2O_3 with Cr^{3+} atoms.



Note that the lasing emission is possible provided that $N_u > N_l$ and:

$\tau_{21} \ll \tau_{10}$:	for a 3-level laser
$\tau_{32}, \tau_{10} \ll \tau_{21}$:	for a 4 level laser

In other words: *non-radiative* emission must be way faster than lasing emission. Note that the 4-level laser can actually work as a 2-levels laser as long as levels 3 and 1 are really close to levels 2 and 0.
An example of 4-levels is $Nd : YAG$, known as neodimium laser.

1.4 Quantum defect/Efficiency

Start observing this following law: $h\nu_{pump} > h\nu_{laser}$
The energy lost is trasformed into lattice vibrations: the temperature increases as result of the non radiative emissions.
Quantum defect:

$$QD = \Delta E = E_{pump} - E_{laser} = h\nu_p - h\nu_L = \left(1 - \frac{\nu_L}{\nu_p}\right)h\nu_p = QD_{\%}h\nu_p$$

Quantum efficiency:

$$QE = \frac{E_L}{E_p} = \frac{\lambda_p}{\lambda_L}\ [\%] = 1 - \frac{\Delta E}{E_p} = 1 - QD_{\%}$$

1.5 Optical Gain

Microscopically :

Optical intensity:	I $\left[\frac{W}{cm^2}\right]$
Population inversion:	$\Delta N = N_2 - N_1 \ \left[cm^{-3}\right]$
Amplification by units of length:	$\frac{dI}{dz} = \sigma(N_2 - N_1)I$

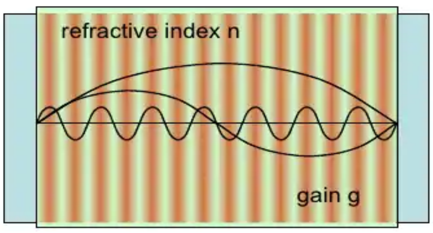
<u>*Macroscopically* :</u>

Integration of I:	$I(l) = I_0 exp(\sigma(N_2 - N_1)l)$
Gain:	$G_{opt} = \frac{I(l)}{I(0)} = exp(gl)$

1.6 Fabry-Perot optical resonators ("etalons")

Reflectivity of the mirror:	$R = \frac{P_r}{P_i}$
Loss-less with zero attenuation:	$P_t = P_i - P_r$
Real relation:	$P_t < P_i - P_r$
Transmittivity:	$T = \frac{P_t}{P_r}$
Reflectivity and Trasmittivity:	$R + T = 1$
In general:	$R + T + A = 1$

Do consider the following scheme representing the oscillation of the generic electromagnetic wave:



Following the path of a "round-trip", it's evident that a wave doesn't disappear only if it respects the condition of constructive interference. For this reason the waves that can survive inside the cavity are those which respect the following mathematical condition:

$$2L = m\lambda \quad m = 0, \pm1, \pm2, \dots$$

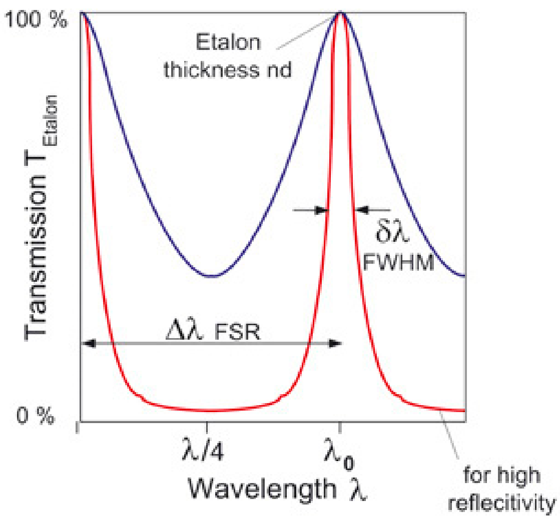
If we look at the picture, the red stripes refer to interference optimals, while the white ones correspond to the nodes.

Resonant condition:	$L = m\lambda/2$
Resonant frequencies:	$\nu_m = m\frac{c}{2L}$
Free Sprectral Range:	$\Delta\nu_{FSR} = \nu_{m+1} - \nu_{mu-1} = \frac{c}{2L}$

Note: the FSR is literally the distance between two peaks.

1.7 Transmission curve

Observe with attention the following picture:



We call *full width at half maximum*, or *transmission linewidth*, the value $\delta\nu$, which represent the "spread" around a resonant frequency of the wave transmission curve. The professor uses the term $\Delta\nu_c$, but "c" is replaced with a e when the cavity id a filter.

Finesse: $$F = \frac{\Delta\nu_{FSR}}{\delta\nu}$$

Round-trip phase difference: $$\varphi = Ks = \frac{2\pi}{\lambda}2L = \cdots = 2\pi\frac{\nu}{\Delta\nu_{FSR}}$$

Transmission explicit expression: $$\frac{P_{out}}{P_{in}} = T = \frac{(1-R_1)(1-R_2)}{1 + R_1R_2 - 2\sqrt{R_1R_2}\cos\varphi}$$

Observing the round-trip phase difference, when $\nu/\Delta\nu_{FSR}$ is an integer the corresponding value of the transmission function is at one maximum. On the other hand, the only one possibiliy to have $T = 0$ occurs for R_1 ord R_2 equal to 1, but it's actually impossible.
Now, we define a unique value for the reflectivity of the oscillator: $R = \sqrt{R_1R_2}$.
Peaks becomes narrower when R increases: the selectivity improves and F becomes higher too.

Finesse : $$F = \frac{\Delta\nu_{FSR}}{\delta\nu} \approx \frac{\pi}{(1-R)}\sqrt{\frac{1+R^2}{2}} \approx \frac{\pi\sqrt{R}}{1-R}$$

Visibility: $$V = \frac{T_{Max} - T_{Min}}{T_{Max} + T_{Min}} \leq 1$$

1.8 More Properties

Loss of photons over time:	$N_{ph} = N_{ph,0} \exp\left(\frac{t}{\tau_c}\right)$
Cavity lifetime:	$\tau_c = \frac{L}{c\gamma}$
Cavity Linewidth:	$\Delta\nu_c = \frac{1}{2\pi\tau_c}$
Quality factor (resonant factor):	$Q = \frac{\nu}{\Delta\nu_c} = \frac{\nu}{\Delta\nu_{FSR}} F$

1.9 Optical Gain (2)

The *threshold* is the starting condition $\underline{after}$ which we can measure an optical gain associated to the beam emitted by the oscillator. Hence, first of all, population inversion must occur inside the active medium. Next, consequently, the threshold is achieved when gain is equal to total losses.

Threshold condition:	$\Delta N_{MIN} \equiv \Delta N_{th} > 0$
Round-trip :	$I_0 \mapsto GI_0 \mapsto R_2GI_0 \mapsto R_2G^2I_0 \mapsto I_0R_1R_2G^2$
Gain equal losses:	$I_0 = I_0R_1R_2G^2$
Threshold gain:	$G_{th} = \frac{1}{R_1R_2} = \frac{1}{R}$

Now, recalling the exponentional definition of gain:

$$exp[2\sigma(N_2 - N_1)l] = \frac{1}{R}$$

Hence:

$$\sigma(N_2 - N_1)l = \frac{1}{2}\Big[-\ln(R_1) - \ln(R_2)\Big] = \gamma$$

We have finally obtained the relation between the threshold condition and the level of pumping, gauged by $N_2 - N_1$:

$$(N_2 - N_1)_{th} = \frac{\gamma}{\sigma l}$$

In the end we define also the slope efficiency as follows:

$$\eta_{slope} = \frac{dP_{laser}}{dP_{pump}}$$

We can logically understand why, at least initially, the higher is the pumping, the higher is the output power. However, at some point the laser would saturate (the losses expression is logarithmic) and the relation becomes less then proportional.

2 A realistic approach

2.1 General concepts

For istance, this means that the pumping can be less precise: an incident radiation at $\lambda_{p,min}$ lets the transition from ground to $E_{3,max}$, while at $\lambda_{p,max}$ transition occurs from ground to $E_{3,min}$.

For what concerns the gain of an active medium, take a look at the gain versus wavelenghts plot. This curve, that looks like a hill, it's actually a distribution function that shows for each value of pumping wavelenght the corresesponding gain obtainable. Do remind that the gain is proportional to the difference in atomic concentrations between the levels involved in the lasing emission.

Now, since we are going to use a cavity of Fabry-Perot, it's important to remember that the oscillations are allowed for specific and periodic values (equally spaced in frequency). These "values" of λ, or ν, are called modes of the cavity. In particular, the trasmission (or the gain of the light emitted) is supposed to be same for every mode.

Which modes survives inside the cavity?
To know that, we must overlay the previous plots. The modes (called longitudinal) for which the trasmission is more the G_{FWHM} but less than G are those wich survive.

How do you filter one resonant frequency/ only one longitudinal mode?

Using a band-pass filter realized by means of another Fabry-Perot cavity, with: $\Delta\nu_e < c/2L_{opt}$ (the cavity linewidth is less than FSR of the longitudinal modes). Moreover, if we have $FSR_E >\approx G_{FWHM}/2$ only one mode is allowed, since the filter singles out only one peak that respect the constraining laws of the two cavities.

2.2 Single-frequency semiconductor lasers

We need very selective mirrors. In the following a brief list of the solutions between we can choose:

- Distributed Bragg Reflector
- Distirbuted Feed-back
- VCSEL: vertical cavity surface emitting laser

2.3 Pulsed Laser

Q-Switchin. Recall that the greater the Q-factor, the shorter the bandwidth, the higher the amplitudes corresponding to admissible frequencies and the lower the losses.

More precisely, the "Q-switch" is an attenuator that is added to the structure and it's exploited for changing the Q-factor at the right istant.
The thecnique of Q-switching starts with the pumping of the active medium with a low Q-factor value. In this situation, since the losses are high, the lasing emission does not occur, but the amount of energy stored in the gain medium increases as it is pumped. Next, when it

gets saturated (there are inevitable losses associated to other processes; for istance, the spontaneous emission), we increase the Q-factor moving the Q-switch. The lasing emission occcurs very rapidly and a very high peak energy pulse of light is emitted.
There are two types of Q-switching laser: the electro-optic and the acousto-optic.

Time travel between repetitions:	T_{REP}, but depends one the Q-switch;
Pulse duration:	$\tau_p \approx 10ns$, depends on the active medium;
Duty cycle:	τ_p/T_{REP}, low duty cycle, high peak power.

Mode locking. It's not requested to explain how it works, or the principle underlying its functionality. However, compared to Q-switching, the mode locking lasers produce much higher pulse repetition rates, much lower pulse duration and greater pulse energies.

Time travel between repetition:	$T_{REP} = \frac{c}{2L}$
Frequency of the repetition:	$f_{REP} = \frac{1}{T_{REP}}$
Pulse duration:	$\tau_p = \frac{1}{B_{laser}} \propto \frac{1}{G_{FWHM}}$
Peak Power:	very high, up to MW, or GW

2.4 Comparison: pulsed Nd:YAG versus Nd:glass lasers

We know that $c = \lambda\nu$.

To find the relation between the variations of wavelength and frequency, we derive over λ and over ν. We obtain:

$$\frac{\Delta\lambda}{\lambda} = -\frac{\Delta\nu}{\nu}$$

For example, fixing $\lambda = 1mm (\nu = 300Thz)$, we compute $\Delta\lambda_{YAG} \approx 0.4nm$, and $\Delta\lambda_{glass} \approx 40 \times \Delta\lambda_{YAG}$.
Consequently, exploiting the relation derived above, we can evaluate the gain badnwidth in terms of frequency:

$$\Delta\nu_{YAG} = 125GHz, \qquad \Delta\nu_{glass} = 5THz$$

Hence, the pulse durations (in mode locking regime) are:

$$\tau_{p,YAG} \approx \frac{1}{\Delta\nu_{YAG}} \approx 8ps, \qquad \tau_{p,glass} \approx \frac{1}{\Delta\nu glass} \approx 200fs$$

The glass laser is more reactive than Nd:YAG laser. However is more fragile. On the other hand, the Nd:YAG laser can emit more powerful pulses, but they last longer.

Moreover, I must mention this further useful relation: $P_{peak}\tau_p = P_{AVE}T$.

3 Laser Source: properties and applications

Note that, sometimes, we have taken the sun (or a generic lamp) as reference to give a numerical idea of these following parameters.

- Main properties of laser source (in general):
  - Monochromaticity ($\Delta\nu_{laser} \approx 10^{-12}\Delta\nu_{sun}$)
  - Brightness ($B_{laser} \gg B_{sun}$)
  - Directionality ($\Omega = \pi\theta^2 \rightarrow \Omega_{laser} \approx 10^{-10}\Omega_{lamp}$)
  - Stability in amplitude ($^{\Delta P}/_{P} \approx 10^{-6}$)and frequency($^{\Delta\nu}/_{\nu} \approx 10^{-15}$)
  - Ultrashort pulses, high peak power
  - Size
  - Propagation: it depends on the medium in which the pulse is travelling
  - Commercial use
- Physical properties of lasers
  - Spatial quality of the beam, known as *spatial coherence*. It's related to the directionality;
  - Spectral quality, known as *temporal coherence*. It's related to the monochromaticity:
  - The color: another name for the wavelength;
  - State of Polarization
  - Optical Power or Pulse Energy

3.1 Properties of Laser Beams

Plotting the optical intensity of the fundamental mode(TEM_{00}), on the trasverse axis, we find out that the electric field distirbution resembles a Gaussian function. The mode is symmetrical in both direction x and y.

Spot size:	ω
Beam waist:	ω_0, where the beam is narrowest
Reference condition:	minimum at $z = 0$, $\omega > \omega_0$ for $z \neq 0$
Electric field distribution:	$E_x = E_0 \exp\left[-\frac{r^2}{\omega_0^2}\right] = E_0 \exp\left[-\frac{x^2+y^2}{\omega_0^2}\right]$
Intensity:	$I = I_0 \exp\left[-2\left(\frac{x^2+y^2}{\omega_0^2}\right)\right]$
Intensity constant:	$I_0 = 2\frac{P_0}{\pi\omega_0^2}$
Optical power:	$P(r) = P_0\left[1 - \exp\left(-2\frac{r^2}{\omega_0^2}\right)\right]$

FREE-SPACE PROPAGATION

Spot broadening (divergence):	$\omega^2 = \omega_0^2 + \left(\frac{\lambda z}{\pi \omega_0}\right)^2$
Rayleigh range:	$z_R = \frac{\pi \omega_0^2}{\lambda}$
Hence:	$\omega = \omega_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2}$

When $z < z_R$ we are in the *near-field region*, and in this region the beam is said to be *collimated*. Besides, we can approximate $\omega \sim \omega_0$, thus $\omega \approx 1.4\omega_0$.

When $z > z_R$ the "enlargement" of the beam becomes important. We call this region *far-field region*, in which the beam is said to be *divergent* (linearly).
Hence, supposing $z \gg z_R$:

$$\omega \approx \frac{\lambda}{\pi \omega_0} z = \theta z$$

And θ is the parameter that gauges the divergence:

$$\theta = \frac{d\omega}{dz} = \frac{\lambda}{\pi \omega_0}$$

In the end we define the M-factor:

$$M^2 = \left(\frac{\theta_{MultiMode}}{\theta_{DL}}\right) > 1$$

3.2 Frequency Noise

The electric field varies with the frequency

$$E(t) = E_0 \exp\{-j[\underbrace{2\pi\nu_0 t + \varphi(t)}_{\phi_{tot}}]\}$$

With ${}^{\Delta\nu}/_{\nu_0} \ll 1$. Let's compute the istantaneous frequency:

$$\nu(t) = ({}^1/_{2\pi})\frac{d\phi_{tot}}{dt} = \nu_0 + ({}^1/_2)\,\frac{d\varphi(t)}{dt} = \nu_0 + \Delta\nu(t)$$

Now, recalling the resonant frequencies definition:

$$\nu = m\frac{c}{2L} \to \Delta\nu = m\frac{c}{2L^2}(-\Delta L) \to \frac{\Delta\nu}{\nu} = -\frac{\Delta L}{L}$$

Hence changing the length of the cavity will affect the fluctuations both in terms of frequency or wavelength.

4 Optical Power

4.1 Introduction

Electric field:	$E = E_0 \exp(-j\omega_0 t)$ $[{}^{V}\!/_{m}]$
Characteristic impedance of vaccum:	$\eta = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\Omega$
Inensity ($\propto$Irradiance):	$I = \frac{E \times E^*}{\eta} = \frac{E_0^2}{\eta_0}$ $[{}^{W}\!/_{m^2}]$
Power:	$P = \int IdS[W] \rightarrow E_0 \propto \sqrt{P_0}$

We use laser to emit a laser beam ad detector to receive it. This detectors can be based on semiconductor technology or be thermally reactive.
In the first case we can choose between two types of device: photo-voltaic or photo-conductive.

Range of values:	$0.1\mu m \leq \lambda \leq 10\mu m$
Quantum efficiency:	$\eta = \frac{\Delta N_e}{\phi \Delta t} = \frac{\#\text{photo-electrons}}{\#\text{incident photons}}$ [%]
Photons flux (microscopic paremeter):	$\phi = \frac{\text{photons}}{S} = \frac{\Delta N_{ph}}{\Delta t}$
Responsivity (macroscopic paremeter):	$\sigma = \frac{i}{P} \left[\frac{A}{W}\right]$
Current:	$i = \frac{e\Delta N_e}{\Delta t}$
Power :	$P = \phi h\nu$
Respositivity 2:	$\sigma = \frac{\eta e}{h\nu}$

4.2 Photodiodes, photodetectors and metohods of detection

PHOTODIODES

Suppose to have a detector with a surface known, S(m^2).
If the beam is uniform, with intensity I, the power received is $P = IS$.
For photodiodes the output is measured in terms of current produced: $i = \sigma P$.

PHOTODETECTORS

The photodiode is typically amplified by a transimpedance gain (resistance) with $G_{i\rightarrow v}({}^{V}\!/_{A}) = R(\Omega)$, in order to produce a voltage output: $v = Ri = G_{i\rightarrow v}\sigma P$
Moreover:

$$v \propto P \propto |E|^2$$

DIRECT DETECTION

One laser beam hits directly the photodetector. We consider the electric field of this beam the following function:

$$E(t) = E_0(1 + \alpha(t))[-j(2\pi\nu_0 t + \phi(t))]$$

Where:

Amplitude [V/m]:	E_0
Amplitude modulation:	$\alpha(t)$
Frequency:	$\nu_0 = 100THz$
Phase/frequency modulation:	$\phi(t)$

What is measured?

$$v(t) \propto EE^* = (E_0)^2[1 + \alpha(t)]^2 \propto P(t) = P_0\alpha_{AM}(t)$$

It's apparent that the all information about frequency/phase is lost.
Besides, usually the $\alpha_{AM} < 1$, since it refers to an attenuation.

BEAT NOTE OF TWO OPTICAL SIGNALS/ COHERENTE DETECTION

Received signal:	$E_R(t) = E_{R0}\exp[-j(2\pi\nu_0 t + \phi(t))]$
Local oscillator:	$E_L(t) = E_{L0}\exp[-j2\phi\nu_L t]$

For semplicity we consider nill the phase/frequency modulation of the local oscillator and both fields linearly-poplarized. Hence, for the superposition theorem: $E(t) = E_R(t) + E_L(t)$

Do consider the beam uniform:

$$P(t) = IS = S\frac{EE^*}{\eta} = \frac{S}{\eta}(E_R E_R^*) + (E_L E_L^*) + (E_R E_L^*) + (E_L E_R^*) = |E|\sqrt{\frac{S}{\eta}} = \sqrt{P}$$

And finally:

$$P_R(t) = P_R + P_L + 2\sqrt{P_R P_l}\cos[2\pi(\nu_R - \nu_L)t + \phi(t)]$$

Beat note:	$\nu_R - \nu_L = \nu_{IF}$
Maximum:	$P_{MAX} = [\sqrt{P_R} + \sqrt{P_L}]^2$
Minimum:	$P_{MIN} = [\sqrt{P_R} - \sqrt{P_L}]^2$

The most particular case: $E_{R0} = E_{L0} = E_0$ and hence $P_R = P_L = P_0$. We can the observe complete interference between the signals. It means that $E_{MIN} = 0$ and $E_{MAX} = 2E_0$, for which $P_{MIN} = 0$ and $P_{MAX} = 4E_0$.

About interference: the fringes of visibility are periodic fluctuations of the power value between its maximum and minimum. We can write that

$$V = \frac{P_{MAX} - P_{MIN}}{P_{MAX} + P_{MIN}}$$

The plot of these fringes change if we change the ratio P_R/P_L

5 Laser alignment and dimensional measurements

To start, we want to exploit te possibility of keeping the beam well collimated, with constant spot size during propagation. In general we want to minimize it in the working region, that has a range of $\pm z$. Hence, we must design an optimal ω_0 at the center of this range.
Once that requirements are satisfied, we can use the beam for practical applications.

Rayleigh range: $$z_R = \frac{\pi\omega_0^2}{\lambda}$$

Spot size: $$\omega(z) = \omega_0\sqrt{1+\left(\frac{z}{z_R}\right)^2}$$

Radius of curvature: $$r(z) = z\left[1+\left(\frac{\pi\omega_0^2}{\lambda z}\right)^2\right] = z\left[1+\left(\frac{z_R}{z}\right)^2\right]$$

What happens when $z \gg z_R$?

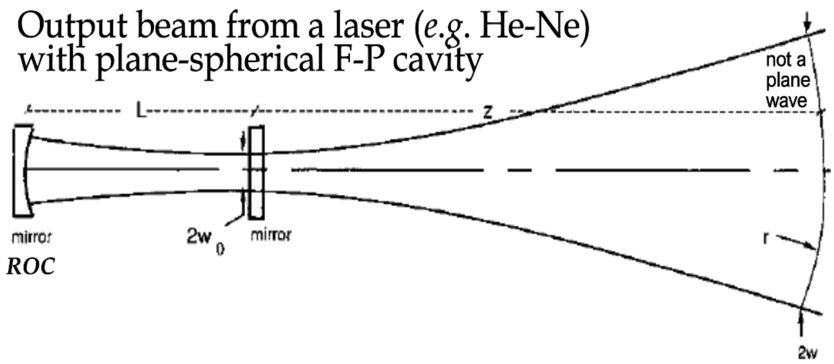
$$\omega(z) \approx \frac{\lambda z}{\pi\omega_0} \approx \theta z$$ the diverging becomes linear

$$r(z) \approx z$$

More precisely when $z = \infty$, the wave is said to be plane. Moreover, when $z = z_R$, then $|r_{min}| = 2z_R$.

The collimation region in which we are interested is defined as follows: from A_{MIN} to $2A_{MIN}$, from ω_0 to $\sqrt{2}\omega_0$, from $r = \infty (z = 0)$ to $r = \pm z_R$.

For a plane-spherical Fabry-Perot cavity there is a specific value which is usually known: the ROC, that is the radius of curvature computed at distance L from the central lens.



$$ROC = r(L) = L\left[1+\left(\frac{\pi\omega_0^2}{\lambda L}\right)^2\right]$$

While L is fixed, ω_0 is not given, but we must evaluate it:

$$\omega_0 = \sqrt{\frac{\lambda L}{\pi}}\left[\frac{ROC}{L}-1\right]^{1/4}$$

For stability $L \ll ROC$.

5.1 Propagation through lens and radius trasformation

Geometrical optics:	$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{f}$
Distances from the central coordinate:	$L_i = z(\omega_i) - z(\omega_{i,0})$
"Thin Lens" condition:	$\omega_1 = \omega_2$
In Far-Field region:	$r_i \approx L_i$

Hence:

$$r_1\theta_1 = r_1\frac{\lambda}{\pi\omega_{1,0}} = \omega_1 \approx \omega_2 = r_2\frac{\lambda}{\pi\omega_{2,0}} = r_2\theta_2$$

And finally:

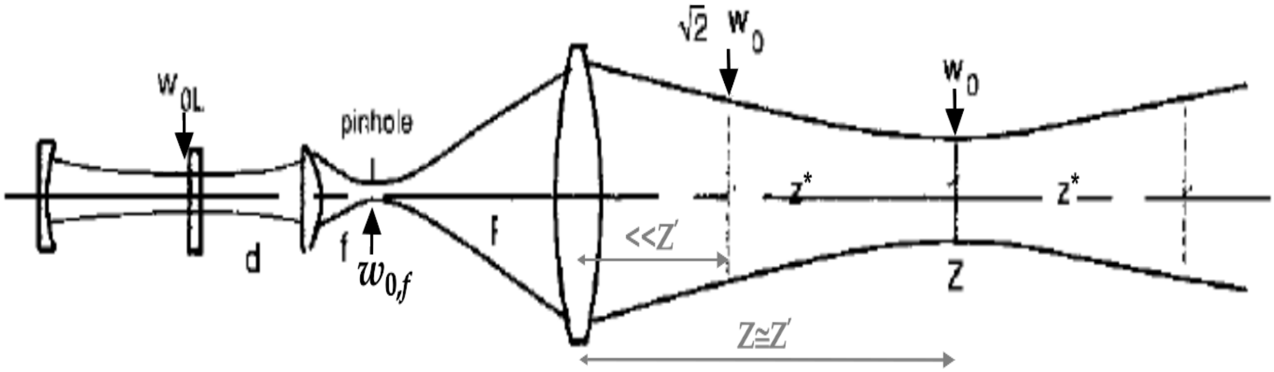
$$\frac{r_1}{\omega_{1,0}} = \frac{r_2}{\omega_{2,0}} \rightarrow \frac{\omega_{1,0}}{\omega_{2,0}} \sim \frac{r_1}{r_2} \sim \frac{L_1}{L_2}$$

Or also:

$$\frac{\omega_{1,0}}{L_1} = \frac{\omega_{2,0}}{L_2}$$

Magnification factor: $m = \frac{\omega_{2,0}}{\omega_{1,0}} = \frac{r_2}{r_1} = \frac{L_2}{L_1}$

5.2 Collimation over a range $\pm z$ through a telescope



As mentioned before, now that we have our scheme, we want to minimize $\omega(z^*)$. Firstly, we suppose to know ω_0 at the center of the range. If we derive $\omega(z^*)^2$ over ω_0^2 (exploiting indeed a substitution) we obtain:

$$\frac{d\omega^2}{d\omega_0^2} = 1 - \left(\frac{z^*}{z_R}\right)^2 = 0 \rightarrow z^* = z_R = \frac{\pi\omega_0^2}{\lambda} \Rightarrow \omega_0 = \sqrt{\frac{\lambda z^*}{\pi}}$$

Note that z* is half-width of the total collimation range.

Relations for the lenses: $\frac{\omega_{OL}}{d} = \frac{\omega_{0,f}}{f}$ and $\frac{\omega_{0,f}}{F} = \frac{\omega_0}{Z}$.

Magnification: $\omega_0 \simeq \frac{Z}{F}\frac{f}{d}\omega_{OL} \Rightarrow m = \frac{\omega_0}{\omega_{OL}} = \frac{Z}{F}\frac{f}{d} = \frac{Z}{d}\frac{1}{M}$

The relative variations: $\frac{\Delta\omega_0}{\omega_0} = \frac{\Delta f}{f}$

To adjust the values of ω_0 and Z it is sufficient to move the lens with short focal length, the one before the "pinhole", provided that $f \ll F$.

5.3 Beam centering on a target and Position-Sensitive photo-detectors

First to know: a photodetector is used to get an electrical signal that is proportional to the alignment. Ths signal is used as feedback and an elctro-mechanical system provides the alignement control.

4-QUADRANTS PHOTODIODE: the surface is typically circular and divided into 4 parts by symmetrical axis. When the light beam hits the detector, a photocurret arises, produced in the depletion region of the p-n junction. We measure a different value for each part. Next, we combine them and we compute the signals referred to the x and y axis, as follows:

X-signal: $S_X = (S_2 + S_4) - (S_1 + S_3)$

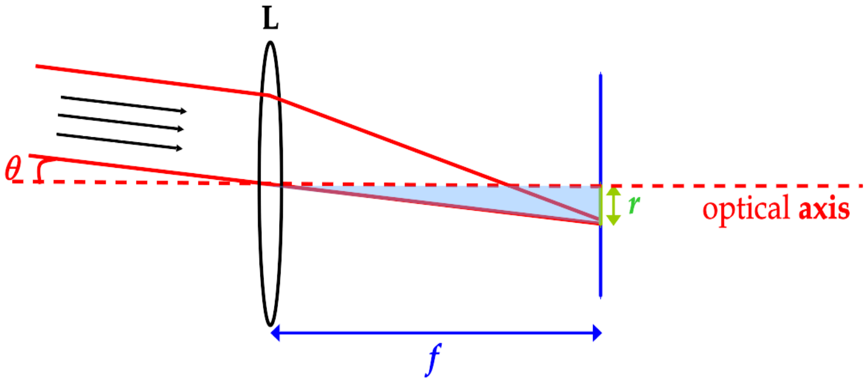
Y-singal: $S_Y = (S_1 + S_2) - (S_3 + S_4)$

Power: $P_0 \propto S_0 = S_1 + S_2 + S_3 + S_4$

PSD PHOTODIIODE: there are two metallic stripes at the ends of the surface, along the x- and y- directions. The superior surface is made of p-doped semiconductor and the metallic stripes are anodes. On the other hand, the inferior surface is made of n-doped semiconductor and the metallic stripes are catodes. In between, the semiconductor is intrinsic.
When the light beam hits the superior surface, a photocurrent arises and it flows from the catodes to the anodes. Differences in current detected on the same electrode pairs gives the corresponding coordinate.

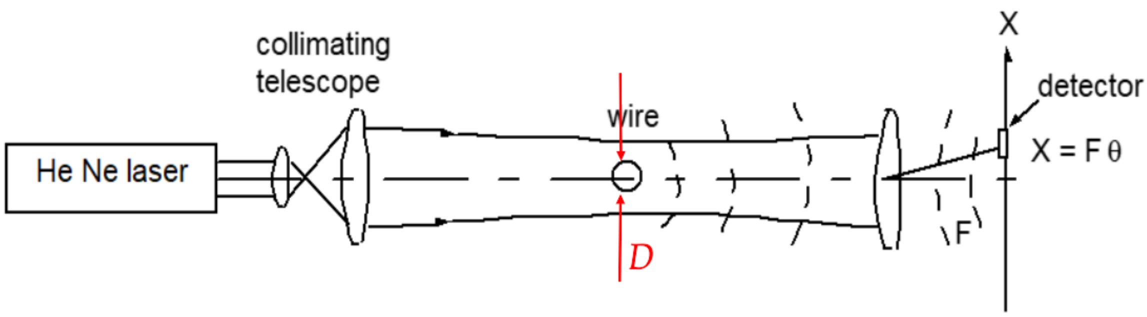
5.4 Examples of Measurements

Preamble: how to trasform an angular unit into spatial coordinate ("position")



In general, $r = f\tan(\theta) \approx f\theta$ for $\theta \ll 1$
Definition of "field of view" : $\theta_{FOV} = {}^{r_{PD}}/_{F}$, with F the focal length of a objective lens ('L' in the picture).

Wire dimeter measurement

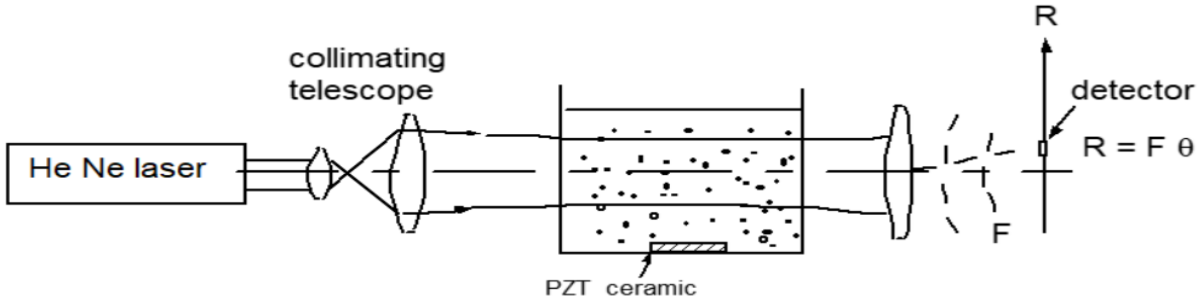


The wave is approximately plane untill it dosn't reach the wire. The interaction with the wire leads to wave diffraction and interference. Do suppose that $D \gg \lambda$. The pattern of the fringes is approximately rectangular. Next, the electric field on the detector is computed by means of the Fourier Transformation of the "aperture" (the rectangular pattern). Besides, remember that the power is proportional to the electric field.

We obtain the $sinc\left(\frac{\pi\theta}{\lambda/D}\right)$.

Angular distribution:	$I(\theta) = \frac{E_0^2}{\eta_0} sinc\left(\frac{\pi\theta}{\theta_D}\right)$
Diffraction angle:	$\theta_{Diff} = \pm{}^{\lambda}/_{D}$
Zeroes on the detector:	$X_{zeroes} = \pm F\theta_D = \pm F{}^{\lambda}/_{D} \Rightarrow D = F{}^{\lambda}/_{X_{zero}}$

Particles diameter measurement



This device is used to measure the diffracted light by suspended patciles whitin a fluid. A photodiode measures the $I(R)$ and inverting its expression the distribution of particles diameter is derived. Differently from the previous example, $D > \lambda$.
The Fourier transformation is referred to the aperture of this case, that is circular:

$$somb\left[\left(\frac{R}{F}\right)\left(\frac{\lambda}{D}\right)\right] = \left[\frac{\theta}{\theta_{Diff}}\right]$$

Suppose to know R, that is the distance of the fringe from the center of the detector ($R = F\theta$), and $p(D)$, that is the probability distribution function of finding a certain particle with dimeter D.
Hence:

$$I(R) = I_0 \int_{0\to\infty} somb^2[(^D/_\lambda)(^R/_F)]p(D)dD$$

6 Optical Telemeters

Definition:*tele-metry* means measuring the distance between the instrument and a remote target, respectively called *laser range-finding* and *laser range-finders.*
There are three methods:

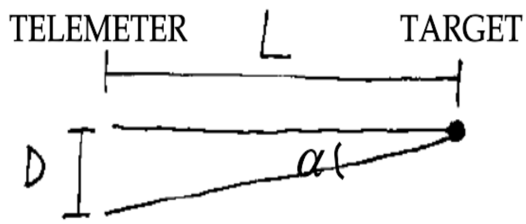
- **triangulation**; that is a trigonometric method, historically apllied to measure the distance to the stars. The target is triangulated from two points apart D ont the same baseline and measuring the angle between the two line of sight. Then $L \approx D/\alpha$, since $\tan\alpha \approx \alpha$.
- **time of flight**; that consists into counting time intervals or phase differences. It's achievable using pulse laser of CW-sine-modulated lasers.

  Time interval: $T = {}^{2L}/_{c} \Rightarrow L = {}^{Tc}/_{2}$

  Phase diff: $\Delta\phi = 2\pi f_m T = 2\pi f_m ({}^{2L}/_{c}) \Rightarrow L = {}^{c}/_{2f_m} {}^{\Delta\phi}/_{2\pi}$
- **interferometry**. This last method consists into counting the number of optical wavelengths, $\Delta\varphi_{optical}$. The laser sent hits the target and comes back to the start, where is coherently detected. The signal goes as $\cos(2kL)$, whit $k = {}^{2\pi}/_{\lambda}$. Actually teh distance is counted in terms of half-wavelenghts.

6.1 Triangulation

General Scheme of an optical passive triangulator



$$\frac{D}{L} = \tan\alpha \approx \alpha \text{ se } \alpha << 1 \Rightarrow L = \frac{D}{\tan\alpha}$$

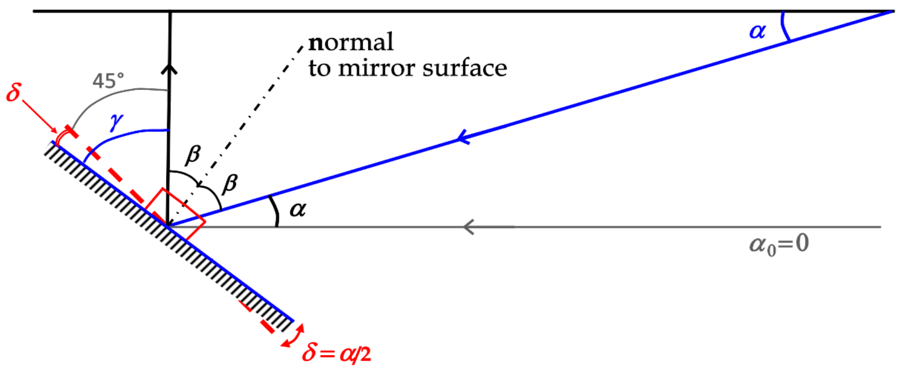
For really long distances the triangulation becomes unrealible. For istance, the absolute error is:

$$\Delta L = -\frac{D}{\alpha^2}\Delta\alpha = -\frac{L^2}{D}\Delta\alpha = -\kappa^2\Delta\alpha$$

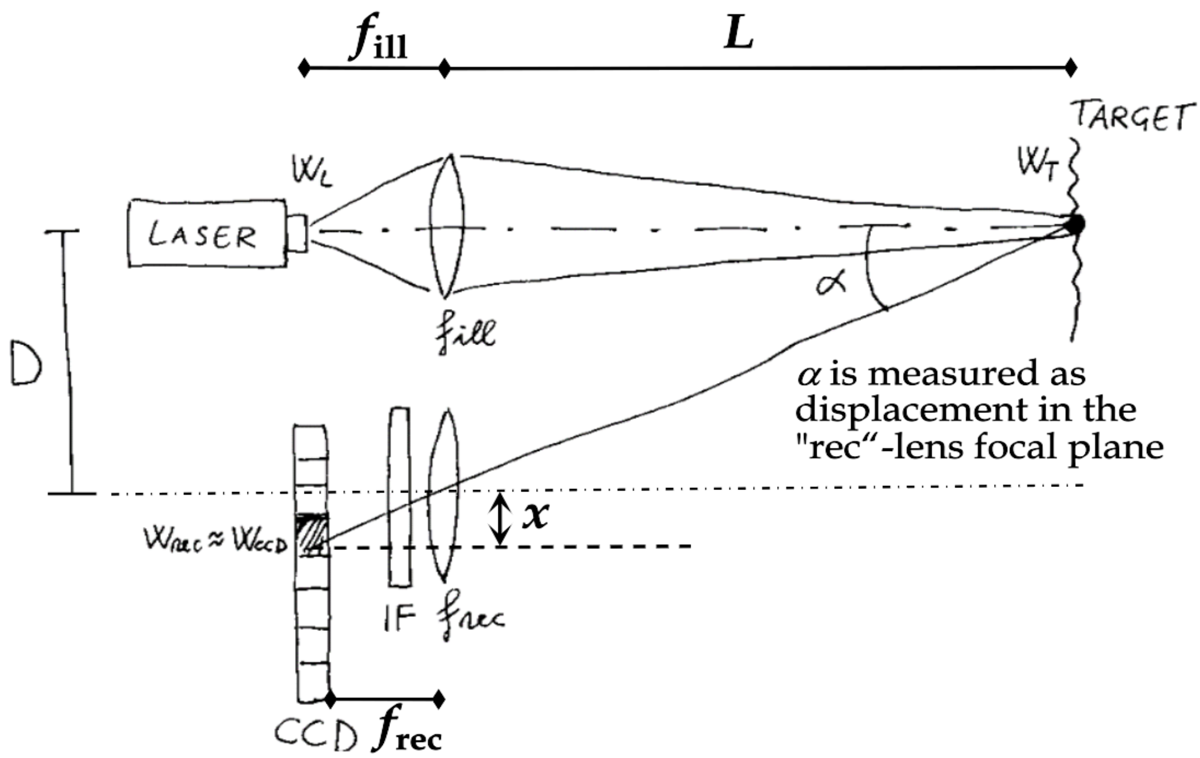
Where κ is the sensitivity. The relative errore is instead defined as follows:

$$\frac{\Delta L}{L} = \frac{\Delta\alpha}{\alpha}$$

Now, if we had placed a mirror at distance D from the telemeter and we send a light pulse (by menas of a laser) to the target, two "rays" come back and combine. To focus on the target we have to rotate the mirror, in order to influence the superposition of the rays.



We can achieve an accurate measurement even with laser and lenses. Furthemore, this type of triangulation is also very repeatable.
The laser beam undergoeas a round trip path, and photodetector measures the angle α between the incident and reflected beam. We use visible λ for semplicity.



- $\tan(\alpha) = {}^{D}/_{L} = {}^{x}/_{f_{rec}}$, for geometric reasons. Consequently $L = ({}^{D}/_{x}) f_{rec} \propto {}^{1}/_{x}$;
- ${}^{D}/_{L}$ is small but not too much, the minimum distance depends on the length of the CCD (the photodetector);
- "IF" is an interference passband filter that removes ambient light.
- Remind that, in the case of a gaussian beam: $\theta_L f_{ill} = \theta_T L$ and $\theta_T L = \theta_{rec} f_{rec}$. Hence:

$$\frac{\omega_L}{f_{ill}} = \frac{\omega_T}{L}, \; \frac{\omega_T}{L} = \frac{\omega_{rec}}{f_{rec}} \Rightarrow \omega_{rec} = \frac{f_{rec}}{f_{ill}} \omega_L$$

Suppose that we change the target position. We would have a variation $\pm\Delta L$.
Since $L = \frac{D}{x} f_{rec}$ because $\alpha = \frac{f_{rec}}{x}$, derive L over x leads to this following relation:

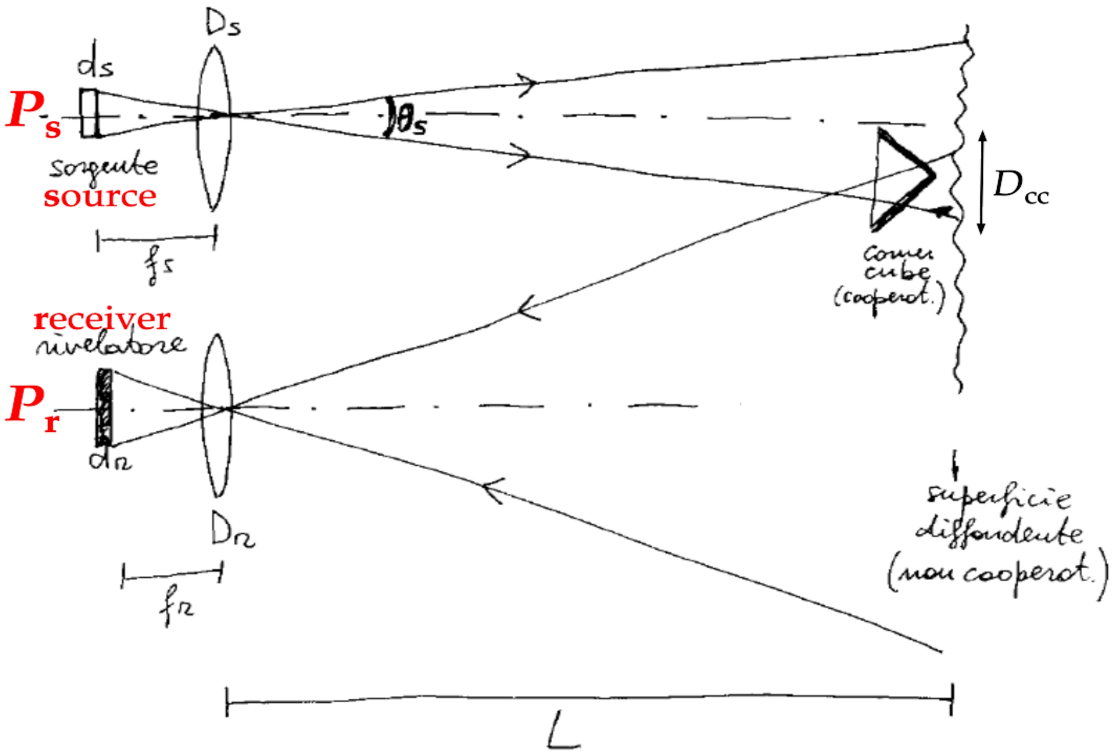
$$\Delta L = -\frac{D}{x^2} f_{rec} \Delta x$$

We don't care about measuring α like with passive triangulators, but we obtain the same equation anyway.
Finally:

$$\frac{\Delta L}{L} = -\frac{\Delta x}{x} = -\frac{\Delta \alpha}{\alpha}$$

.

6.2 Power Budget in optical telemeters



Above all, $L \gg f_s, f_r, D_s, D_r$. Secondly, the vertical surface is defined *non-cooperative*, its diffusity is less than one: $\delta < 1$. On the other hand, the corner cube is cooperative, $R \approx 1$.

The divergence angle is: $\theta_s = \frac{d_s}{f_s}$

As for cooperative surfaces, they behave like a mirror: if we sent a beam against a vertical mirror it eventually returns, but the spot size will be twice as large. Moreover, the receveir will see the source at distance $2L$. The corresponding beam spot size (diameter) is $\theta_s 2L$.
Hence, a quality parameter involving the power is the ration between received power and power from the source.

$$\frac{P_r}{P_s} = \frac{(\pi/4)D_r^2}{(\pi/4)\theta_s^2 4L^2} < 1$$

It's computed as the ratio between the receiving area and the receiving beam area.

- Case 1: the corner cube is smaller than spot size, $D_{cc} < \theta_s L$, but the receiver collects all reflected light (unrealistic). As we will see, it's computed as the areas ratio at corner cube. Suppose that $D_r = 2D_{cc}$:

$$\frac{P_r}{P_s} = \frac{D_r^2}{\theta_s^2 L^2} = \frac{4D_{cc}^2}{4\theta_s^2 L^2} = \frac{D_{cc}^2}{\theta_s^2 L^2}$$

- Case 2: if, in addition to the corner cube, even the receiving lens is shorter than reflected beam spot size, we came back to the formula introduced before. Th ratio between received and emitted power is the areas ratio at the receiver. We can say then, both "mirror" and receving lens are cutting the beam. The more realistic condition is when $D_r < 2D_{cc}$.

Atmospheric attenuation.

In reality the laser undergoeas absorption and scattering losses due to molecules and particulate alawys present in the atmosphere. Mathematically, a rigorous expression is:

$$\frac{P(z=2L)}{P(z=0)} = \exp(-\alpha 2L) = T_{atm} \quad \textit{(Lambert-Beer Law)}$$

Where α is the attenuation coefficient, $\alpha(\lambda) = a(\lambda) + s(\lambda)$.

Atmosphere	coefficient
Very Clear	$\alpha = 0.1 km^{-1}$
Clear	$\alpha = 0.3 km^{-1}$
Little	$\alpha = 0.5 km^{-1}$
Foggy	$\alpha \gg 0.5 km^{-1}$

When we have to take into account the atmospheric attenuation: $\dfrac{P_r}{P_s} = T_{atm}\dfrac{D_r^2}{\theta_s^2 4L^2}$

Equivalent Length: $$L_{eq} = \frac{L}{\sqrt{T_{atm}}} = L\exp(\alpha L) > L$$

Telemeter gain: $$G_c = \frac{1}{\theta_s^2}, \quad \text{with "C" the cooperative target}$$

Telemeter Power budget: $$\frac{P_r}{P_s} = G_c \frac{D_r^2}{4L_{eq}^2}$$

6.3 Solid Angle, Brightness, Lambert Emitter

SOLID ANGLE

Arc:	"r"
Radius of the arc:	"R"
Radian (plane angle):	$\theta = \frac{r}{R}$
Cap Area:	"S"
Steradian (solid angle):	$\Omega = \frac{S}{R^2}$

The solid angle is computed by means of a proportion: $\Omega : \Omega_{tot} = S : S_{tot}$.

Hence, considering a sphere: $\Omega = 4\pi \frac{S}{S_{sfera}} = 4\pi \frac{\pi r^2}{4\pi R^2} = \frac{r^2}{R^2}\pi = \pi\theta^2$

RADIANCE, OR BRIGHTNESS

Suppose to have a surface emitting radiation.

Infinitesimal surface:	dS
Normal direction:	$\vec{n}$
Direction of interest:	$\vec{r}$
Angle between n and r:	θ
Visibile surface:	$dS_{vis} = \vec{n}\vec{r}dS = dS\cos(\theta)$
Power emitted by dS:	$dP = L(d\Omega dS\cos(\theta))$
Radiance/Brightness:	L

The brightness is the ratio of power per unit solid angle, per unit surface in the direction of view.

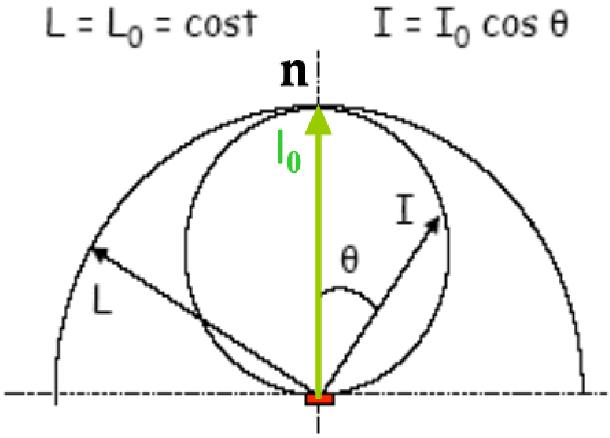
$$L = B = \frac{dP}{d\Omega dS\cos(\theta)}$$

LAMBERT EMITTER

The radiated light depends on the cosine function of the angle referred to the normal direction

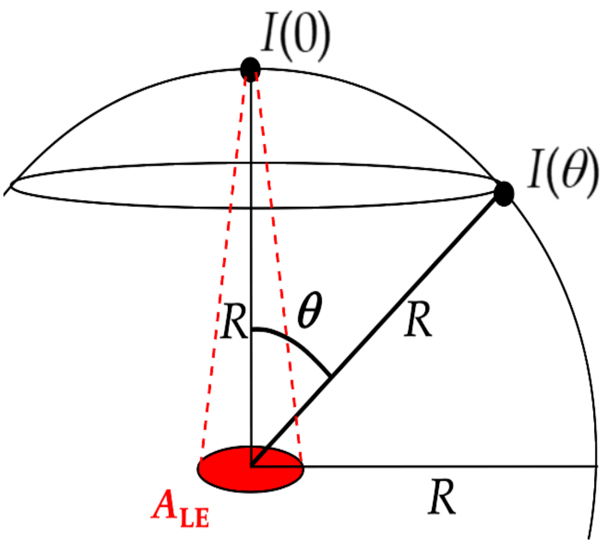
Maximum:	$I_{MAX} = I_0$, along the normal direction ($\theta = 0$)
Minimum:	$I_{MIN} = 0$, along direction $\theta = \pm 90$
Intensity:	$I = I_0\cos(\theta)$
Far from the source (R » "source size"):	$dP/dS \approx P/S = I$
Brightness:	$B = L = I_0/d\Omega = constant$

It is evident that B is constant with θ.
In the following picture the polar diagrams of B and I:



Note that the diagram of B is a half of a sphere, i.e. the external one.

Now, we want to compute the power emitted in the superior half space. To indicate Lambert Emitter we use the acronym LE:



Intensity:	$I(\theta) = I_0 \cos(\theta) = B_{LE}\dfrac{A_{LE}}{R^2}\cos(\theta)$
Infinitesimal area:	$dA = 2\pi[Rsin(\theta)][Rd\theta]$
Power emitted:	$P_{LE} = \int_A I(\theta)dA = \int_0^{\pi/2}(\dots)d\theta = B_{LE}A_{LE}\pi$

The "equivalent" solid angle is finally defined, with respect to the previous result: $\Omega_{eq} = \pi$

When we work with a non-cooperative target, its illuminated surface, with area A_T, is diffusing light with a diffusion coefficient $\delta < 1$. Suppose that this target respects the Lambert characterization (Lambertian diffuser):

$$B_{LE} = \frac{I}{\pi} = \frac{\delta P_s}{\pi A_T}$$

In general, with a non-cooperative diffusing target...

Fiel of view of the receiver: $\theta_r = \frac{D_r}{2L}$

Receiver solid angle: $\Omega_r = \pi\theta_r = \frac{\pi D_r^2}{4L^2}$

These parameters are used to evaluate by which angle the target is seen by the receiver. The power collected by the receiver is:

$$P_r = B_T A_T \Omega_r = \frac{\delta P_s}{\pi A_T} A_t \frac{\pi D_r^2}{4L^2} = \delta \frac{D_r^2}{4L^2} P_s$$

In the end, to compute the POWER BUDGET we obtain:

$$\frac{P_r}{P_s} = \delta \frac{D_r^2}{4L^2}$$

It's evident that the power received is independent from the area on the target and the power budget resembles the formula of a cooperative target with δ instead of $1/\theta_s^2$.

6.4 To summarize...

Considering losses due to Tx and Rx optics ($T_{opt} \leq 1$) and round trip propagation in the atmosphere ($T_{atm} \leq 1$):

Cooperative target: $\left[\frac{P_r}{P_s}\right]_C = T_{optic,C}\, T_{atm,C}\, \frac{D_r^2}{\theta_s^2 4L^2}$

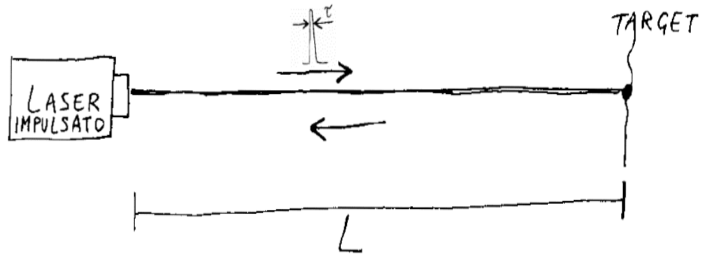
Non cooperative target: $\left[\frac{P_r}{P_s}\right]_{NC} = T_{optic,NC}\, T_{atm,C}\, \delta \frac{D_r^2}{4L^2}$

Gain: $G_c = \frac{T_{optics,c}}{\theta_s^2}$ or instead $G_{nc} = T_{optics,nc}\delta$

Equivalent Field Of View: $FOV_{EQ} = \frac{D_r}{2L}$

Very General expression : $\left[\frac{P_r}{P_s}\right] = G \frac{D_r^2}{4L_{eq}^2} = G\, FOV_{EQ}^2$

6.5 Time of Flight Telemeters



$$T = \frac{2L}{c} \Rightarrow L = \frac{c}{2}T$$

- Variations: $\Delta L = \frac{c}{2}\Delta T \Rightarrow \frac{\Delta L}{L} = \frac{\Delta T}{T}$
- Length resolution ΔL of the measurement depends on the time resolution, ΔT and so on the pulse duration τ.
- Time resolution measures the possibility to locate the pulse, on the time axis. We need a τ smaller the ΔT. This requirement is satisfied if we work with fast photodetector electronics: $B \approx \frac{1}{\tau}$.
- In general we can measure the time interval by means of an electronic counter, associated to a frequency of clock. It counts the number of clock cycles starting at the time instant at which the laser signal is emitted and it ends the counting when the returning signal is photodetected. Then, since T doesn't likely fall exactly on clock, we have to measure the uncertainty.

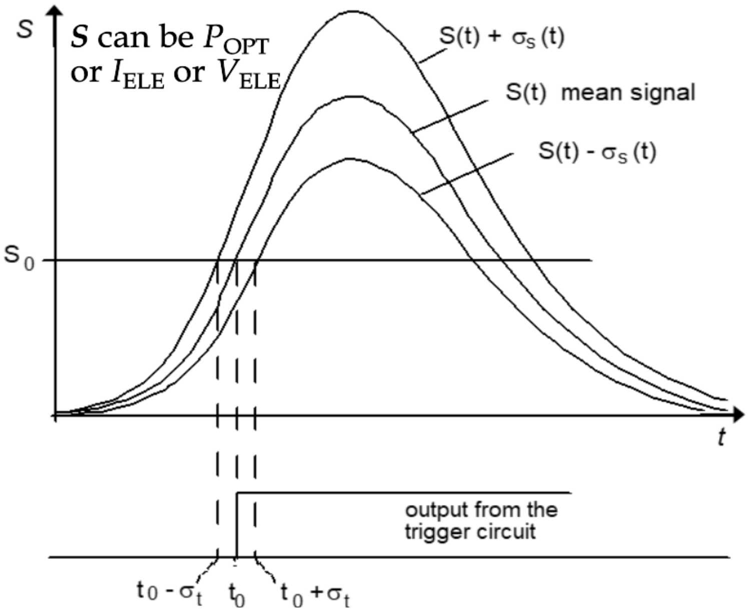
Time interval T:	$T = t_{start} - t_{stop} \approx NT_c$
Time interval of the clock:	$T_c = \frac{1}{f_c}$
Uncertainty:	$u_q(t) = \sigma(t) = \frac{T_c}{\sqrt{12}} \approx 0.3T_c \approx T_c$
Total uncertainty:	$u(T) = \sqrt{u_{start}(t)^2 + u_{end}(t)^2}$
Clock synchronous with start:	$u(T) = u(t_{stop}) = \frac{T_c}{\sqrt{12}}$

Choosing T_c short enough the uncertainty will depend majorly on the overall amplitude noise, sum of the electric circuit noise and ampltidue noise of the detected optical signal. We've defined the TOF as follows: $T = t_{stop} - t_{start}$. The variance of this parameter, caused by the noise, is:

$$\sigma^2(t) = \sigma^2(t_{start}) + \sigma^2(t_{stop}) \approx \sigma^2(t_{stop})$$

The latter approximation is logical since the light pulses returning back from the target are less powerful than emitted ones.

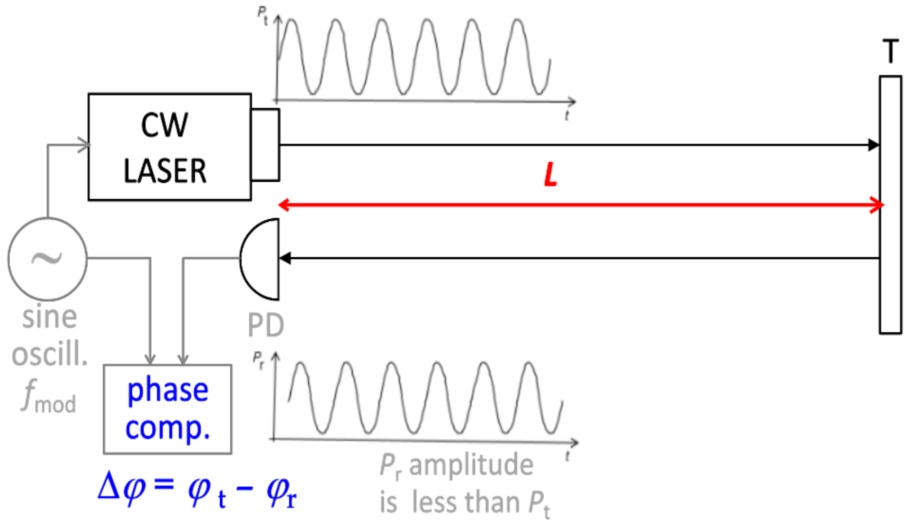
We are able to determine the location of the pulses on the time axis thanks to a trigger circuit. This circuit "takes note" of the time istant at which the output of the photodetector (optical power, irradiance or voltage) is greater than a reference value, in general referred as S_0.



As we can see from the picture, the signal variation σ_s is converted into time variation σ_t by means of the slope at trigger point. Great slope $\rightarrow \sigma_t$ is small.

Look at the following subsection to know something more about the *ambiguity* and how to deal with it in the case of a pulsed laser source.

6.6 CW sine-modulated telemeters



Time of flight: $T = \frac{2L}{c} = \Delta T$

Optical power (modulated) : $P(t) = P_0[1 + m\sin(2\pi f_{mod} t)]$

Delays proportion: $\frac{\Delta\varphi}{2\pi} = \frac{\Delta T}{T_{mod}}$

Period: $T_{mod} = \frac{1}{f_{mod}}$

Phase difference: $\Delta\varphi = 2\pi f_{mod}\Delta T = 2\pi\frac{1}{T_{mod}}\Delta T = 2\pi\frac{1}{T_{mod}}\frac{2L}{c}$

Length: $L = \frac{c}{2}\frac{\Delta\varphi}{2\pi f_{mod}}$

Sensitivity: $S = \frac{\Delta\varphi}{L} \propto L$

The sensitivity tells us how $\Delta\varphi$ changes with a change in the distance. By increasing the frequency we can increase the sensitivity, but when f_{mod} is too high we can occur in measurement ambiguity.
This *ambiguity* concerns the task of distinguishing targets at different distances which may return the same information. To avoid this problem:

1. With pulsed telemeter:
$$T_{NA} = T(L_{NA}) \leq T_{rep}$$
Where T_{NA} is the maximu time of flight, corresponding to the maximum distance L_{NA} correctly measured **before ambiguity**.

2. CW sine-modulated telemeter:
$$\varphi_{NA} = \varphi(L_{NA}) = 2\pi f_{mod} T_{NA} \leq 2\pi \Rightarrow f_{mod} \leq \frac{1}{T_{MAX}}$$
In the end we derive that $T_{NA} \leq \frac{1}{f_{telem}}$, where f_{telem} is a combination of f_{rep} and f_{mod}. Note that the condition on the phase difference is derived from the fact that in the worst scenario ($L = L_{MAX}$) the phase shift must be less than 2π.

6.7 System equation, telemeter SNR, equivalent Power, SNL

Total power noise: $P_n = P_r + < shot, noise > + < electronic, noise >$

SNR: $S/N = \frac{P_r}{P_n}$

Telemeter equivalent power: $P_{eq} = GP_s = \frac{4L_{eq}^2}{D_r^2} P_r = \frac{4L_{eq}^2}{D_r^2} \frac{S}{N} P_n$

Once we've fixed some parameter, like S/N and D_r, lets plot P_{eq} versus L_{eq} on bilogarithmic scales. We will see the typical ranges to work with for typical values of P_n and with respect to the tecnology exploited (pulsed laser or CW).

The causes of the "optical" noise:

- noise associated to the received signal($P_{n,s}$)
- noise associatd to background light($P_{n,bg}$)
- noise of photodetector and transimpedance amplfier("front end", $P_{n,el}$)

$$\Rightarrow P_n = P_{n,s} + P_{n,bg} + P_{n,el}$$

Now, defining I as the DC/signal current: $I_r = \sigma P_r$ is the useful signal current, and $I_{bg} = \sigma P_{bg}$ is the backgroud contribution. In total $I_{rec} = I_r + I_{bg}$. We will use the symbol $I_{el,0}$ to refer to the "virtual" equivalent DC noise current.
For what concerns the AC fluctuations:

1. shot noise on $I_r \rightarrow i_r^2 = 2\,e\,I_r\,B \rightarrow i_{n,s}$
2. shot noise on $I_{bg} \rightarrow i_{bg}^2 = 2\,e\,I_{bg}\,B \rightarrow i_{n,bg}$
3. electronic noise: $i_{el}^2 = 2\,I_{el,0}\,B \rightarrow i_{n,el}$

Since these are thre uncorrelated quantities, the global variance, or total power, of current noise at PD is:

$$i_{n,rec}^2 = i_{n,s}^2 + i_{n,bg}^2 + i_{n,el}^2 \Rightarrow i_{n,rec} = 2eB(I_r + I_{bg} + I_{el,0})$$

Next, dividing all these electrical photocurrents by the squaredd spectral responsivity (σ^2) we get the optical power noise at the receiver input:

$$P_n^2 = \frac{i_{n,rec}^2}{\sigma^2} \Rightarrow P_n = \frac{2h\nu}{\eta} B(P_r + P_{bg} + P_{el,0})$$

This last quantity is the variance of the optical power, or overall noise power.

Background Light

It is not very clear what is "scene", with respect to the concept of background. Look at the following formulas to have an idea of how these terms are used.

Geometrical analysis of background light:	$P_{bg} = B_{sc}\Omega_{sc}A_r$
Brightness (Lambertian diffuser):	$B = \frac{\delta I_{sc}}{\pi}$
Spectral irradiance of solar light:	$E_{sc} \quad (W/m^2\mu m)$
Optical intensity of the scene:	$I_{sc} = E_{sc}\Delta\lambda$
Optical intensity at the receiver:	$I_{bg} = \frac{1}{\pi}[\delta_{sc}I_{sc}]\Omega_{sc}$
Numerical Aperture:	$NA = \sin\left(\frac{D_r}{2f}\right)$
Solid angle:	$\Omega = \pi\theta^2 \approx \pi(NA)^2$
Area of the receiver:	$A = \frac{\pi d_r^2}{4}$
Background power:	$P_{bg} = [\delta_{sc}E_{sc}\Delta\lambda NA^2]\left(\frac{\pi d_r^2}{4}\right)$

SNL

SNL is the acronym for shot-noise limited.

We know the relation betwen Signal and time axis for laser pulsed telemeter, and we've seen that we can convert the signal variation into time variation by means of the slope:

$$\sigma_t^2 = \frac{\sigma_s^2}{[dS/dT]^2}$$

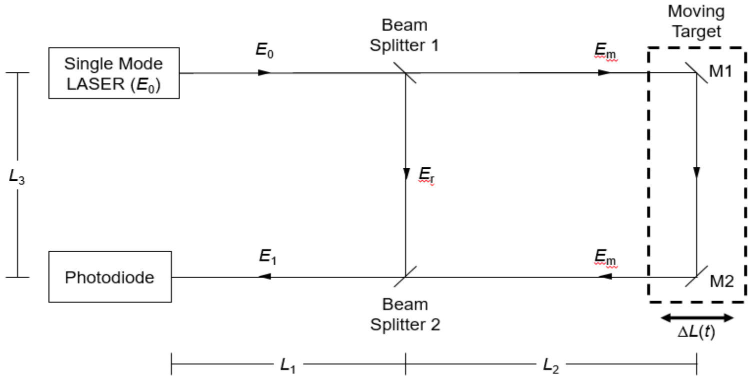
Now, in SNL condition, the background light and electronic noise do not affect the total power at the receiver. Hence, $\sigma_t \propto \frac{\tau}{\sqrt{N_r}}$, where N_r is the number of photons that reach the receiver.

With a CW sine modulated telemeter, we obtain that $\sigma_t = \frac{1}{2\pi f_m}\frac{1}{\sqrt{N_r}}$

7 Interferometers

7.1 Working Principle

Generic Scheme:



We call *reference path* the direct connection between the beam splitters, while we call *measurement path* the other one.
The reference path is always kept constant. On the contrary, the measurement path can change: we move the mirrors, at fixed distance in between, by $\Delta L(t)$. The measurement path increases by $2\Delta L$
Before the first beam splitter and after the second one the electric field are summed togheter. At the output we can see the pattern of their interference.
To evaluate the output of the photodiode, the photocurrent I_{ph}, we must take into account the phase difference between the paths:

$$I_{ph} = \sigma|E_1|^2 = \sigma|E_r + E_m|^2 = \sigma|E_r \exp(j\phi_r) + E_m \exp(j\phi_m)|^2$$

$$\Rightarrow I_{ph} = \sigma[E_r^2 + E_m^2 + 2E_m E_r Re\{\exp(\phi_m - \phi_r)\}] = I_m + I_r + 2(I_m I_r)^{1/2}\cos(\phi_m - \phi_r)$$

Next we define $I_{average} = I_{AVE} = I_0 = I_m + I_r$ and the expresion of the photocurent becomes:

$$I_{ph} = I_0\{1 + \left[\frac{2(I_m I_r)^{1/2}}{I_m + I_r}\right]\cos(\phi_m - \phi_r)\}$$

Besides, focusing on the phase difference:

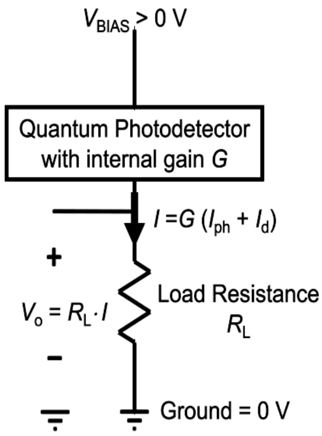
$$\phi_m - \phi_r = k[L_m + 2\Delta L(t)] - kL_r = k[L_m + \Delta L(t) - L_r] = \frac{2\pi}{\lambda}[L_m + \Delta L(t) - L_r]$$

Observations:

- remind that the power is proportional to the photo detected current. Hence $P(t) = P_m + P_r + 2\sqrt{P_m P_r} cos(\varphi(t))$
- if $E_m = E_r \rightarrow I_{MIN} = 0$ and $I_{MAX} = 2I_0$
- if $\phi_m - \phi_r = 2\pi \leftrightarrow \Delta L = \lambda/2$. This means that if we move the mirrors by $\lambda/2$ we find the previous or the following fringe of interference.
- if we measure the derivate signal, we observe 2 counts per interferometric fringe.

These generic scheme and relations are the same caracterizing the homodyne coherent detection setup. From chapter 4 we know that a heterodyne coherent detection is referred to two signals that interfere with each other, with a beat note $\nu_R - \nu_L$. "Homodyne" means that these two frequencies are equal.
Thanks to all these properties we can always work at the *quantum-limit regime of detection*: the signal to noise ratio is optimum and it depends only on the signal level, improving for increasing signal level.

Quantum-limit regime of detection



Current photo-generated:	I_{ph}
Current generated in the "dark":	I_d
Signal :	$S = GI_{ph}$
Quantum Noise:	$QN = QN_{I_{ph}} + QN_{I_d} = 2eI_{ph}G^2FB + 2eI_dG^2FB$
Thermal Noise:	$TN = \dfrac{4k_BTB}{R_L}$
Total Noise:	$N = QN + TN$

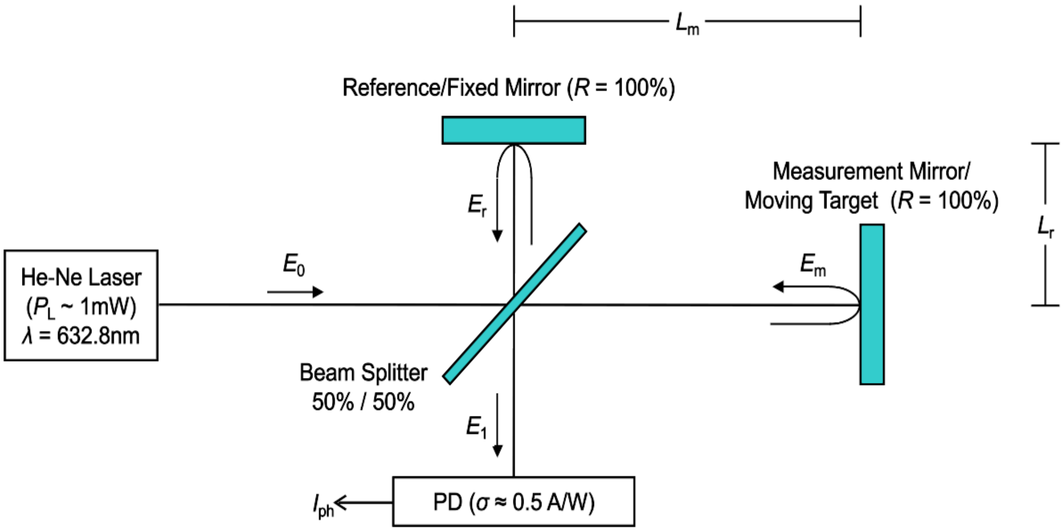
Observations:

- e is the electronic charge, B is the observation bandwidth, F is the excess noise factor. $F = 1$ for an ideal amplification process, otherwise is less greater than 1, also by far.
- if $I_{ph} \gg I_d + \dfrac{2K_BTB}{e}\dfrac{1}{R_LG^2F}$ (or, equivalently: $QN_{I_{ph}} >> QN_{I_d} + TN$) the signal to noise ratio can be written as follows:

$$(S/N)_{rms} = \frac{I_{ph}G}{\sqrt{2eI_{ph}G^2FB}} = \sqrt{\frac{I_{ph}}{2eFB}}$$

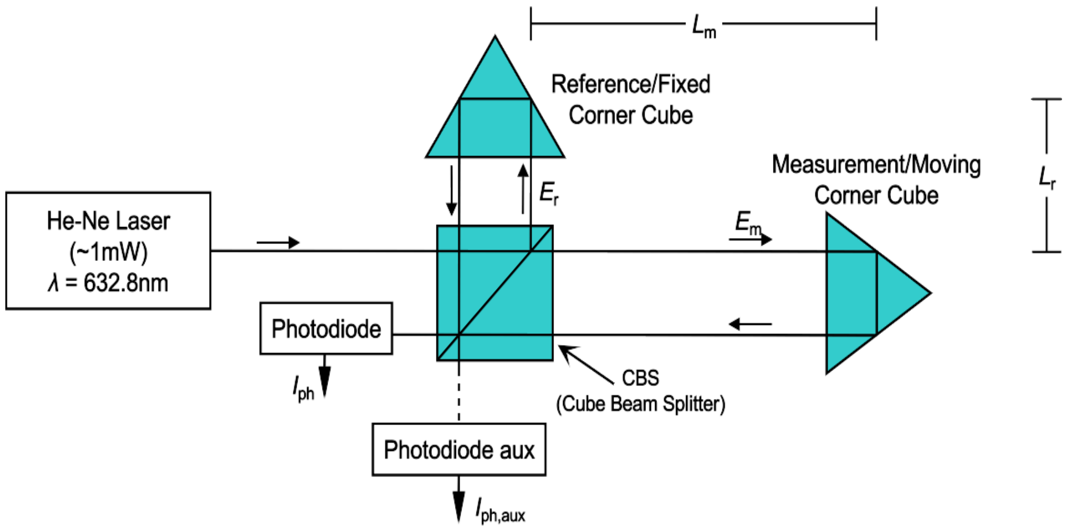
The detection circuit is said to work at the quantum-limit regime or Shot-Noise Limited (SNL). As mentioned before, S/N depends majorly on the photocurrent and bandwtdh, not on the type of detector used, nor G, nor R_L.

7.2 Michelson Interferometer



$$I_{ph} = I_0[1 + \cos(2k(L_m - L_r))]$$

Where $I_0 = 2I_m = 2I_r = {}^1\!/_2\sigma P_L$.
How it works? Well, moving the mirrors we can change the distance L_m. If $2k(L_m - L_r) = 2\pi$ we met the next interferometric fringe. Thus, we have a measurement of the difference between the distances by counting how many fringes we see appearing. This means that the resolution is $\Delta\lambda/2$, if the distance is less than this value we have not a number of fringes to convert into spatial lenght.
However, ther are two issues with this setup: the light is reflected back into the cavity and the alignment of the mirrors is a very critical operation.
The solution consists into replacing the normal mirrors and splitters with corner cubes and corner beam splitter (CBS). There is no longer reflection in the optical cavity and these devices are much easier to align. Moreover, the reference corner cube can be attached directly to the CBS.



7.3 Ambiguity of the cosine signal: $\lambda/4$ plates and setup solutions

With respect to the previous image, we can understand that another issue arises when we want to measure the moving direction of the target (mirrors).
We know that $I_{ph} \approx 1 + \cos(2k(L_m - L_r))$ while $I_{ph,aux} \approx 1 - \cos(2k(L_m - L_r))$. Since the cosine function is even, when the argumet is the same we cannot understand if the target is coming closer or moving farther. We've developed two different setup to avoid this problem, but firstly we must get use to the concept of $\lambda/4$ plates.

$\lambda/4$ -Waveplate

- Waveplates are made of birefringe materials, most commonly quartz. This material have optical properties which depends on the polarization of the light, i.e. they have slightly different indexes of **refraction** for light polarized in different directions;
- A waveplate has a slow axis and a fast axis, both perpendicular to the direction of the beam and also to each other. Light polarized along the fast axis experiences a lower index of refraction and travels through the waveplate faster than light polarized along the slow axis:

$$n_{slow} > n_{fast} \Rightarrow v_{slow} = {}^{c}/_{n_{slow}} < {}^{c}/_{n_{fast}} = v_{fast}$$

- For a smilar reason, the optical path that light sees is different if the beam is "fast-polarized" or "slow-polarized": $L_{opt,slow} = n_{slow}L > Ln_{faster} = L_{opt,fast}$.

We exploit this technology to obtain two laser beams (at the output) which present a well-given phase difference:

$$\Delta\phi = kL(n_{slow} - n_{fast}) = const.$$

Common values are π or $\pi/2$.
The fist one corresponds to the so called "half-wave plates":

$$\Delta\phi = \frac{2\pi}{\lambda}L(n_{slow} - n_{fast}) = \pi \Rightarrow L(n_{slow} - n_{fast}) = {}^{\lambda}/_{2}$$

The second one to the "quarter-wave plates": $L(n_{slow} - n_{fast}) = {}^{\lambda}/_{4}$
The waveplates are used to trasnform the laser beams inside an interferometer, hence the light will not have the same polarization state at the input and at the output.

Double-beam Interferometer

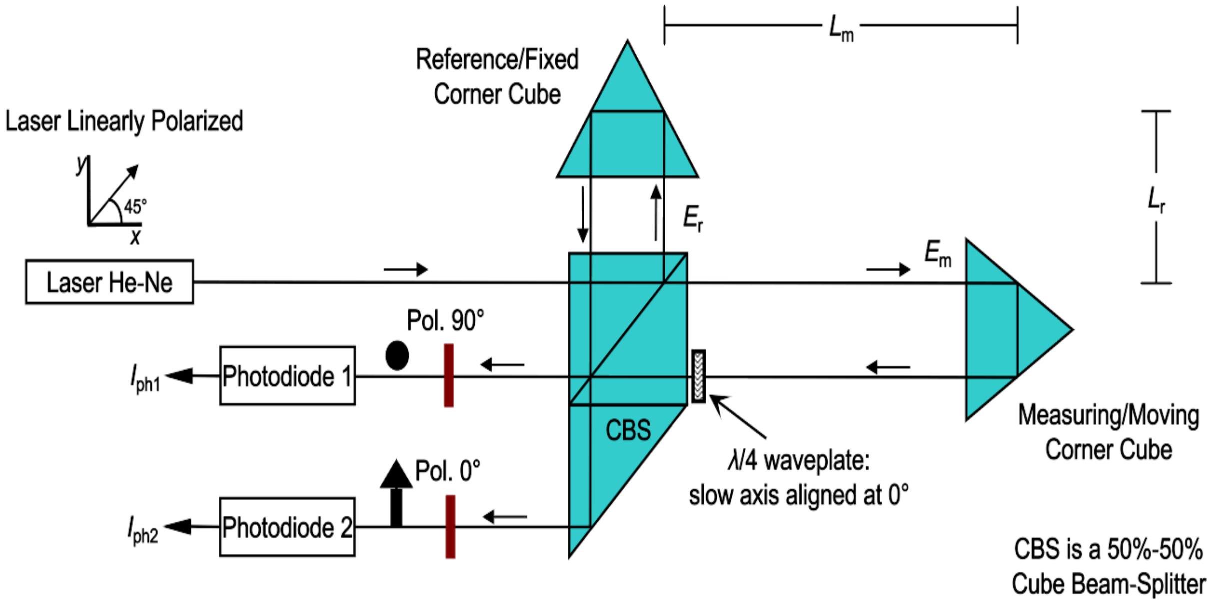
With rescpect to the following picture:

$$I_{ph,1} = I_m + I_r + 2(I_m I_r)^{1/2}\cos(2k(L_m - L_r)) = I_0[1 + \cos(2K(L_m - L_r))]$$

$$I_{ph,2} = I_m + I_r + 2(I_m I_r)^{1/2}\cos(2k(L_m - L_r) + k\lambda/4) = I_0[1 - \sin(2k(L_m - L_r))]$$

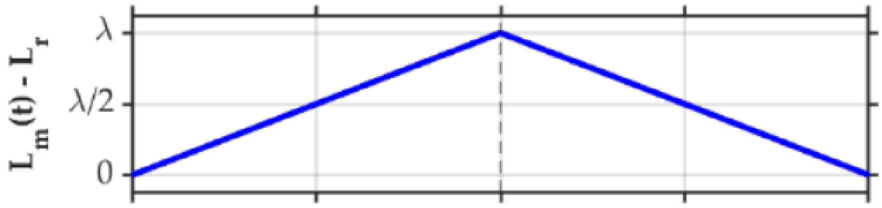
$$I_0 = I_m = I_r = {}^{1}/_{4}\sigma P_L$$

As we can see, $I_{ph,2}$ is modified by the $\lambda/4$ waveplate.

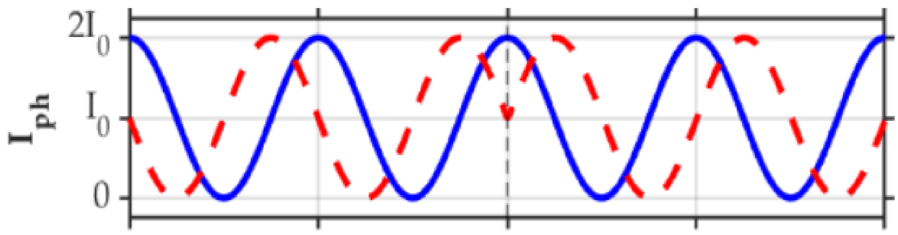


Signal Analysis:

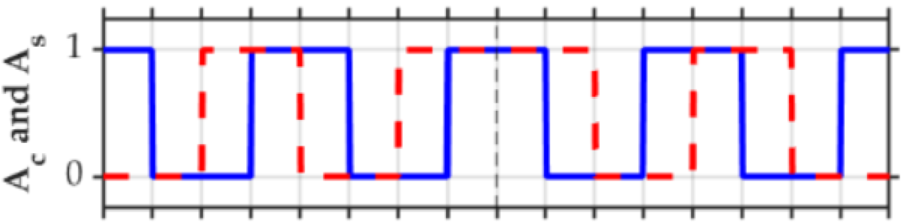
- The difference $L_m(t) - L_r$ varies with time:



- The signals are squared (blu ↔ cos and red ↔ sin) ...

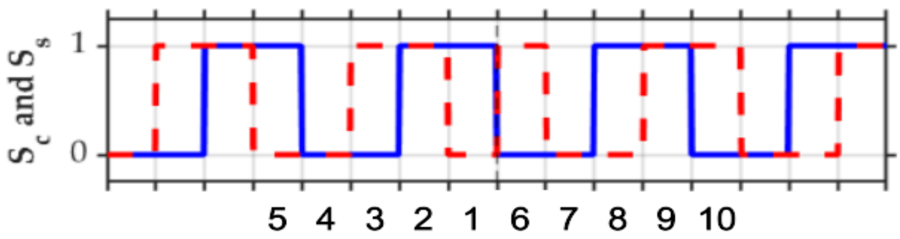


- ... to obtain digital/squared amplitude signals A_s and A_c:



[LOGIC: IF $(A > 0) \rightarrow 1$, $(A < 0) \rightarrow 0$]

- The same operation is done with the derivates of sine and cosine signals, and we get digital/squared slope-signals S_s and S_c:



[LOGIC: IF $SLOPE > 0 \rightarrow 1, SLOPE < 0 \rightarrow 0$]

- Now, what happens if the target changes its position? From the above diagrams we can determine which direction is following the target by looking at how S_c and S_s change: while A_s and A_c are even, the logic values of S_s and S_c are complementary:

	A_c	A_s	S_c	S_s
1	1	1	1	0
2	0	1	1	1
3	0	0	0	1
4	1	0	0	0
5	1	1	1	0

	A_c	A_s	S_c	S_s
6	1	1	0	1
7	0	1	0	0
8	0	0	1	0
9	1	0	1	1
10	1	1	0	1

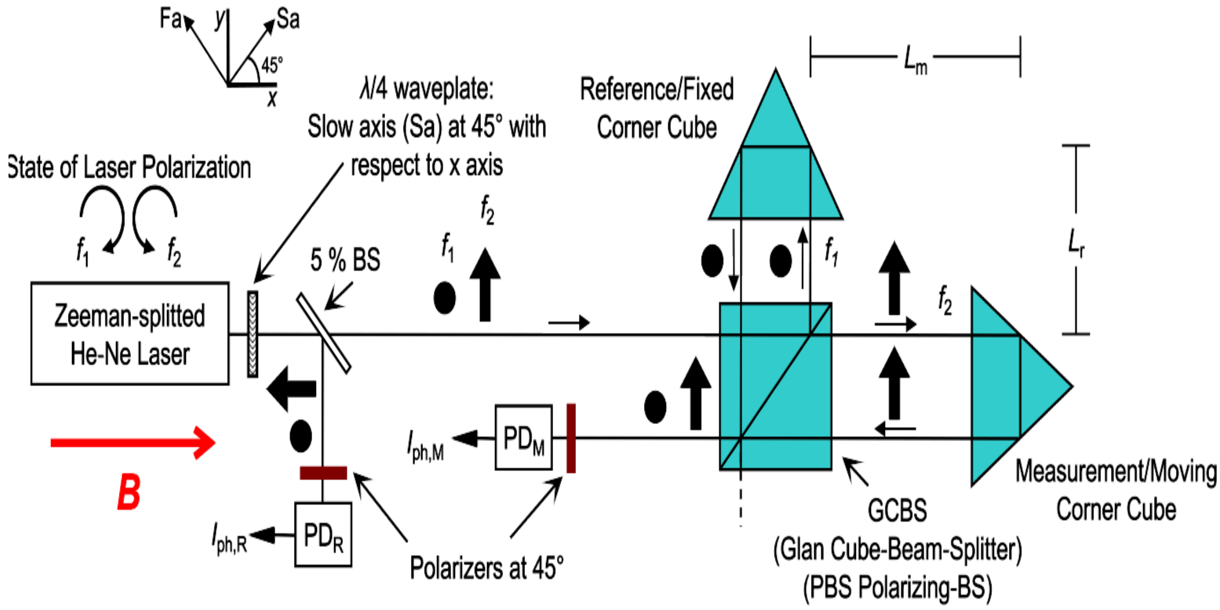
Note that 1,2,3,... are time instants. Hence, the time istants of the table one are in succesion following the axis in the opposite direction. This is useful only to understand that values are "symmetric", obviously we cannot go back in time.

- Since we have two signal, then 4 jumps each fringe, the resolution goes down to $\lambda/4$. The condition to determine one fringe (F) is: $\Delta\phi_F = 2\pi$ or $\Delta L_F = \lambda/2$. We find each pulse moving the target by $\Delta L_{pulses} = \lambda/8$ (it's sufficient to replace λ with the resolution expression).
- In the end we use a specific signal, know as U, which tells us the direction of motion:

$U = 1$: sign '+', the target is moving farther
$U = 0$: signa '-', the target is coming closer

There are several issues about the use of a double beam interferometer. I cannot use it in DC or at a very low frequency. Once we've fixed the baseband B we must respect the limit of its applicability: $\tau_{pulse} = \Delta L_{pulse}/v_{max} \rightarrow v_{max} \leq B\Delta L_{pulse}$. If the otpcial beam is interrupted, the counts are lost and the measurement shall be repeated. EMI, low-frequency Noise and enviromental vibrations can cause spurious counts. In the end, the threshold I_0 of the ampltidue-squaring-comparator is difficult to determine.

Double-frequency Interferometer



- The Zeeman laser emits on two longitudinal modes, at optical frequencies f1 and f2. Each of them is cirularly polarized and orthogonal to the direction of propagation. Obtained applying the magnetic field B at the optical cavity.
- The inteferometric phase at PD_M, that is $2k(L_m - L_r)$, is superposed to a carrier of electrical frequency, $f_c = |f_1 - f_2|$, that is the beat note of the two optical frequencies. We must keep in mind tha the movement of the target is a modulation of the carrier, thus what is the maximum frequency by which you can modulate a carrier? $f_c + B$. Moreover, even if B can be a very high value, we choose a suitable value, e.g. if $f_c \approx 5MHz$, $B = 1MHz$. We can say that the choosen electrical sideband B sets a limit to the frequency.
- Photocurrent of reference: $I_{ph,R} = \frac{5}{100} I_0\{1 + \cos[2\pi(f_1 - f_2)t + \varphi]\}$
- Measurement Photocurrent:
$$I_{ph,M} = \frac{95}{100} I_0\{1+\cos[2\pi(f_1-f_2)t+2kL_m-2kL_r+\varphi]\} = \frac{95}{100} I_0\{1+\cos[2\pi(f_1-f_2)t+2kL_m+\varphi_t\}$$
- As always : $I_0 = 1/2\sigma P_L$. Where σ is the photodiode responsivity, and P_L is the total optical power emitted by the laser.

How to measure $\varphi_m = 2KL_m$?
The new approach consist into digital counting of the zero-level crossings, with positive (or negative) slope, of $I_{ph,r}$ and $I_{ph,M}$ signals, both having a frequency $f_c \approx 5MHz$. The digital countner behaves like an integrator, and if we recall that frequency is defines as $f = \frac{1}{2\pi}\frac{d\varphi}{dt}$:

$$C_R = \int_{[0,T]} (f_1 - f_2)dt = (f_1 - f_2)T$$

$$C_M = \int_{[0,T]} [(f_1 - f_2) + \frac{2k}{2\pi}\frac{dL_m}{dt}]dt = \int_{[0,T]} [(f_1 - f_2) + 2v_m/\lambda]dt = (f_1 - f_2)T + 2\Delta L_m/\lambda$$

...where $v_m = \dfrac{dL_m}{dt}$, T is the refresh time of the measurement ΔL_m is the overall target displacement in a time interval T.

Next, the subtractor in the ciruit evaluates the parameter $S = C_m - C_r = \Delta L_m/(\lambda/2)$.
The resolution is $\Delta L = \lambda/2$ (or $\Delta L = \lambda/4$ if we count the semiperiod).
Thus, we can say that the result is the number of half-wavelengths which correspond to how far the target has been moved. The maximum measurable velocity is:

$$v_{max} = (\lambda/2)B \approx 0.3m/s$$

Pros of the double-frequency interferometer:

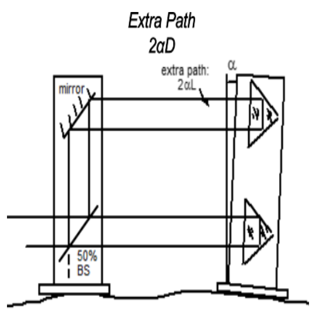
- the threshold value of the comparator is now set at zero, after an high-pass (AC) filtering of $I_{ph,R}$ and $I_{ph,M}$, without loss of information. This is not possible with the double-beam interferometer.
- the rejection of EMI and Low-frequency noise is better at $f \approx 5MHz$.

7.4 Planirity Measurement and Angle Measurement

When α is different from 0 we find an optical path variation of $2\alpha D$, in the upper arm of the intererometer.

$$\Delta L = \lambda/2 \rightarrow \Delta\alpha = \Delta L/2D$$

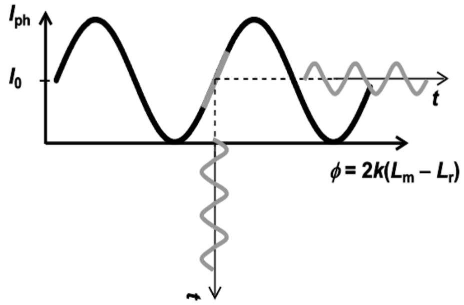
Analogously with a resolution of $\lambda/4$.



7.5 Performance Limits

Half-Fringe Working Point: HFWP

As we've seen in the previous chapters, with interferometers we can measure path variations $L_m - L_r \ll \lambda$. To achieve this goal we have to stabilize the intererometer working point to the so-called "Half-fringe working point", where the system reaches its maximum sensitivity.



$$I_{ph} = I_0\{1 + \cos[2k(L_{m,0} + L_m(t) - L_r)]\}$$

If $2kL_R = \pi/2 + m_1 2\pi$:

$$I_{ph} = I_0\{1 + \sin[2k(L_{m,0} + L_m(t))]\}$$

If $2kL_m(0) = m_2 2\pi$:

$$I_{ph} = I_0\{1 + \sin[2kL_m(t)]\}$$

If $2kL_m(t) \ll \pi/2$:

$$I_{ph} = I_0\{1 + 2kL_m(t)\}$$

And we define the parameter $I_{IS} = 2I_0 k L_m(t)$.
Next, the minimum measurable displacement is called noise equivalent displacement (NED) and it depends on the phase noise of the laser source (NEP_p) and on the electrical noise, minimum, superimposed on the interferometric signal (NED_Q).

Temporal Coherence and Phase Noise

In reality, electromagnetic waves don't behave like an ideal monochromatic signal. There are jumps, which are randomly distributed in time. The mean interval between two jumps is called temporal coherence $\tau_c =< t_{jump+1} - t_{jump} >$.

Definitions:

Laser Linewidth:	$\Delta\nu = {}^1/_{\pi\tau_c}$
Coherence Length:	$L_c = c\tau_c = \lambda^2/\pi\Delta\lambda$

Typical Values:

He-Ne (excellent) :	$L_c \approx 300m$ and $\Delta\nu = 300MHz$, with $\tau_c = 1\mu s$
Diode laser (good) :	$L_c \approx 30m$ and $\Delta\nu = 3MHz$, with $\tau_c = 0.1\mu s$
Diode laser (normal) :	$L_c \approx 1.5m$ and $\Delta\nu = 60MHz$, with $\tau_c = 5ns$

Fringe Visibility

- The maximum displacement is $2|L_m - L_r|$. If $2|L_m - L_r| \gg L_c$, then we cannot perform an interferometric measures **as the signal reduces to a random noise signal**.
- $I_{ph} = I_0[1 + V\cos[2k(L_m - L_r)]$. The parameter V is known as fringe visibility: $0 < V < 1$.
- If we consider a single-longitudinal laser source, with a Lorentzian linewidth, we get that $V = \exp[-\frac{|L_m - L_r|}{L_c}]$.

Quantum-Limit of detection

- if $2|L_m - L_r| < L_c$ we can perform an interferometric meaurement.
- A lower limit exists and it depends on τ_c: higher is τ_c and lower is the limit. Remember that before the minimum measurable displacement has been defined as the Noise Equivalent Displacement, hence NED is the lower limit.
- NED_{phase} (Noise Equivalent Displacement):
  - $\Delta\nu > 0Hz \rightarrow$ the lasing frequency is not constant but randomly changes in time: $\nu(t) = \nu_0 + \Delta\nu(t)$
  - The phase will be time-dependent:

$$\phi(t) = 2k(L_m - L_r) = \frac{4\pi}{\lambda}(L_m - L_r) = \frac{4\pi\nu}{c}(L_m - L_r) = \frac{4\pi}{c}(L_m - L_r)[\nu_0 + \Delta\nu(t)] = \phi_0 + \Delta\phi(t)$$

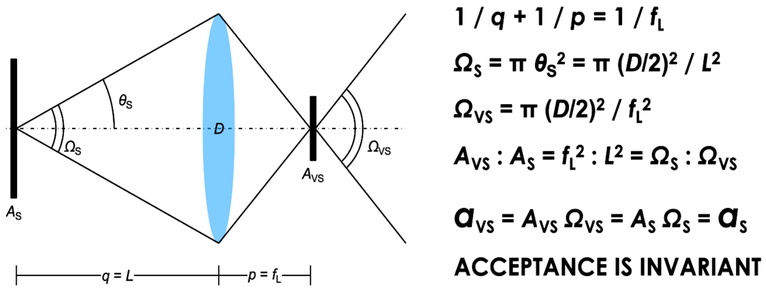
  - Hence: $\Delta\phi = \frac{4\pi}{c}(L_m - L_r)\Delta\nu = \frac{4\pi}{\lambda_0}(L_m - L_r)\frac{\Delta\nu}{\nu_0}$
  - Finally NED_p is defined as: $NED_p = \Delta L_{rms} = \frac{\Delta\phi_{rms}}{2k} = (L_m - L_r)\frac{\Delta\nu}{\nu_0} = \frac{\lambda}{\pi}\frac{(L_m - L_r)}{L_c}$
- Consequently, if $L_m = L_r$ there is not a limitation on the measurement, since $NED_p = 0$. In this case the interferometer is said to be balanced.
- As a consequence of the Measurement and Reference Signal superposition that we can achieve in laser interferometers, we always work with a homodyne coherent scheme of detection. For this reason, it is always possible to work at the QUANTUM LIMIT OF DETECTION. The noise of the system is dominated by the contribution related to the photo-current I_{ph} and if we consider a quantum detector without internal gain and $F = 1$, we have: $i_n^2 = 2eI_{ph}B + 2eI_dB + \frac{4k_BTB}{R_L} \approx 2eI_{ph}B$.
- For what concerns the quantum noise:
  - $I_{ph} = I_0\{1 + V\cos[2k(L_{m,0} + \Delta L_m - L_r)]\}$
  - Signal current at HFWP : $I_{IS} = (I_0V)2k\Delta L_m$

- Power signal to noise ratio: $(S/N)^2 = i_s^2/i_n^2 = \dfrac{(I_0 V 2kL_m)^2}{2eI_0B}$
- The NED_Q is calculated for $(S/N)^2 = 1$, solved for $\Delta L_M = NED_Q$
- $NED_Q = \dfrac{\lambda}{2\pi}\dfrac{1}{V}\left(\dfrac{eB}{2I_0}\right)^{1/2}$
- In terms of interferometric phase, we can write:
  $\Delta\phi_{RMS} = 2kNED_Q = \dfrac{2}{V}\left(\dfrac{eB}{2I_0}\right)^{1/2} = \dfrac{2}{V}\left(\dfrac{h\nu B}{2\eta P_0}\right)^{1/2}$
- Remind that P_0 is the optical mean power, where $P_0 = \sigma_{PD}/i_0$ and $\sigma_{PD} = \dfrac{\eta e}{h\nu}$

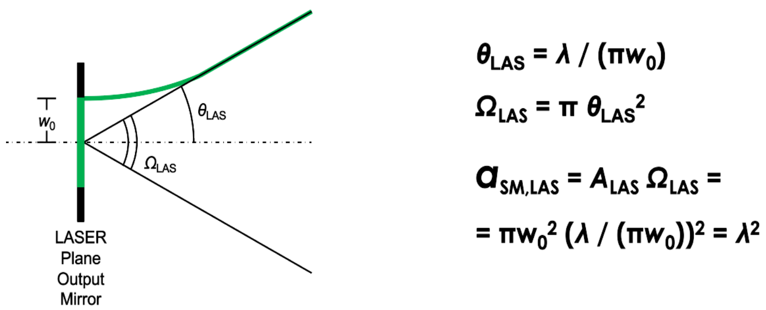
7.6 Acceptance and Radiance (Brightness)

The *acceptance* of a source is defined as $\mathbf{a} = A\Omega$, measured in $[m^2 sr]$. A is the area of emission and Ω is the corresponding solid angle. The physical meaning of **a** is associated to the number of spatial modes carried by the emitted light.

Virtual source and virtual acceptance.

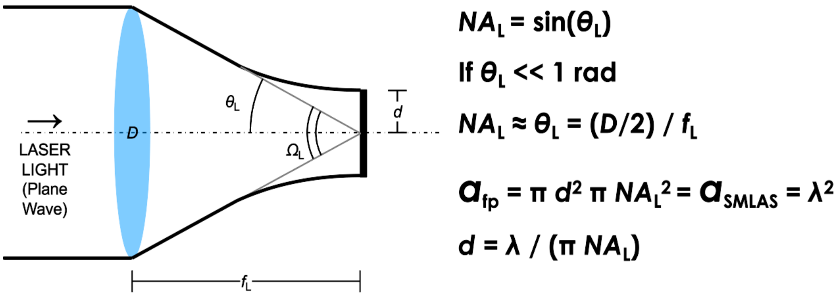


Next, more precisely, for a signle-mode:



Where w_0 is the laser beam waist, Ω_{LAS} is the laser solid angle of emission and θ_{LAS} is the divergence angle.

Numerical aperture (NA_L) of a lens and minimum size of a Single Mode Laser in the focal plane (the minimum "spread").



Obviously D is the diamter of the lens, while f_L is the focal length.

The radiance can hence be defined as: $B = P/\mathbf{a}$, measured in $[W \times m^{-2} \times sr^{-1}]$

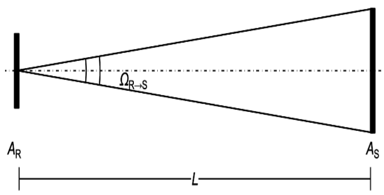
In the case of the virtual source: $B_s = P_s/\mathbf{a}$, thus $B_{vs} = P_{vs}/\mathbf{a}_{vs}$. If all the power emitted by the source is collected by the lens, then we can write that $B_{vs} = B_s$.

The result is valid in general, if there is no absorption/parzialization or scattering in the trasceiving system, radiance is conserved. If the opposite is true: $B_{vs} = B_s$.

Now, when radiance is conserved, the power *at the receiver* in an optical system can be estimated as: $P_R = B_s\mathbf{a}_R$, i.e. the radiance of the source times the acceptance of the receiver. Moreover:

$$\mathbf{a}_R = A_R\Omega_{R\to s} \qquad \Omega_{R\to s} = A_s/L$$

$\Omega_{R\to s}$ is the solid angle given by the source as seen by the receiver.



Suppose to work with a *lambertian diffuser*, that is an ideal case. To calculate the radiances at different angles we use the radiance at $\theta = 0$:

$$E_0 = B_s\Omega_0 = B_s\frac{A_s}{R^2}.$$

$$E_\theta = B_s\Omega_\theta = B_s\frac{A_s\cos(\theta)}{R^2}$$

$$E_\theta = E_0\cos(\theta) = E_{MAX}\cos(\theta)$$

For what concerns the power emitted by the semipherical/Labertian diffuser: $P = B_sA_s\pi$.

7.7 Speckle-pattern

In the most realistic case the target is non cooperative. Hence, if a single mode laser imprenges over a non cooperative diffusing target, the back-scattered beam is no longer "single mode" and we have a typical granular intensity distribution called "speckle-pattern".

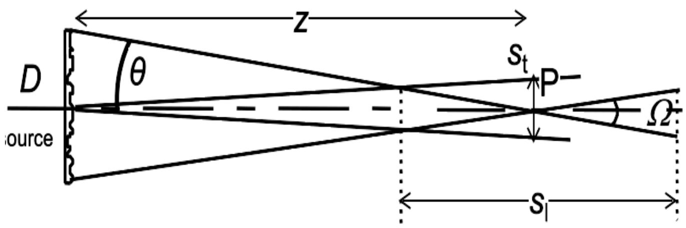
The diffuser can be seen as a surface with roughness larger than the wavelength $\Delta z \gg \lambda$. The field at a generic point P is the sum of many vectors with random phase. Moving from P to $P + \Delta P$, the field gradually loses correlation. We follow a statistical approach to study it.

Intensity of the field:	J
Mean value:	J_M
Density function (of the intensity):	$\rho(J) = {}^1/_{J_M} \exp(-{}^J/_{J_M})$

The probability to have an intensity lower than J_M is: $P\{J < J_M\} = 1 - e^{-1} \approx 63,2\%$.
It's for this reason that is called "speckle": the field is divided in flecks of random intensity. We're facing an intensity-fading problem.

<u>Speckle dimensions</u>

- Trasversal and longitudinal dimensions of the speckle are statistical variables. We can anyway estimare their mean (or expected) values.
- For a circular diffusing target with diameter D, the full statistical approach gives these results: $s_t = \lambda({}^z/_D)$ and $s_l = \lambda({}^{2z}/_D)^2$. It's logical that $s_l \gg s_t$.
- The projections of the out-of-axis speckles are equal to the dimensions of the speckles along the reference axis (z).
- Each speckle is a *single spatial mode* area, thus it has an acceptance. Remind that $\mathbf{a} = A\Omega = \lambda^2$.



Solid Angle:	$\Omega = \pi \left[\frac{(D/2)}{z}\right]^2$
Area (transverse):	$\pi \left(\frac{s_t}{2}\right)^2$
Single mode condition:	$\lambda^2 = \pi^2 \left(\frac{s_t}{2}\right)^2 \left(\frac{D}{2z}\right)^2$
Dimensions:	$s_t = \frac{4}{\pi}\frac{\lambda z}{D} \qquad s_l = \frac{s_t}{\theta} \approx \frac{2}{\pi}\lambda\left(\frac{2z}{D}\right)^2$

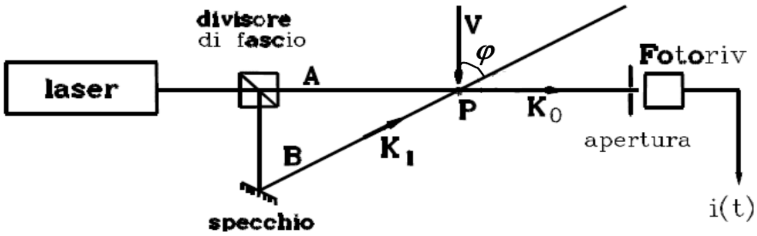
8 Optical Velocimeters

8.1 Measurement Principles

Doppler effect on laser beams.

The target moves away:	$\nu_{observed} = \left(1 - \frac{v}{c}\right) \nu_{laser}$
The target gets closer:	$\nu_{observed} = \left(1 + \frac{v}{c}\right) \nu_{laser}$
Doppler shift:	$\Delta\nu = (\nu_{observed} - \nu_{laser}) = \pm \left(\frac{v}{c}\right) \nu_{laser} \ll \nu$

What happens when the direction of the beam and the velocity of the particle are not the same? We use the projection of $\vec{v}$ on $\vec{k}$. With $\vec{k}$ we mean the propagation vector and with φ the angle between the two directions. Hence $\Delta\nu = v\cos\varphi\frac{v}{c}$.
Note that the shift is zero if $\vec{v}\perp\vec{k}$
Since the value of the shift is very small with respect to the magnitude of the optical frequencies, we cannot use a monochromator, or a OSA. We need to measure instead the heterodyne beat signal with a reference laser beam (unshifted).



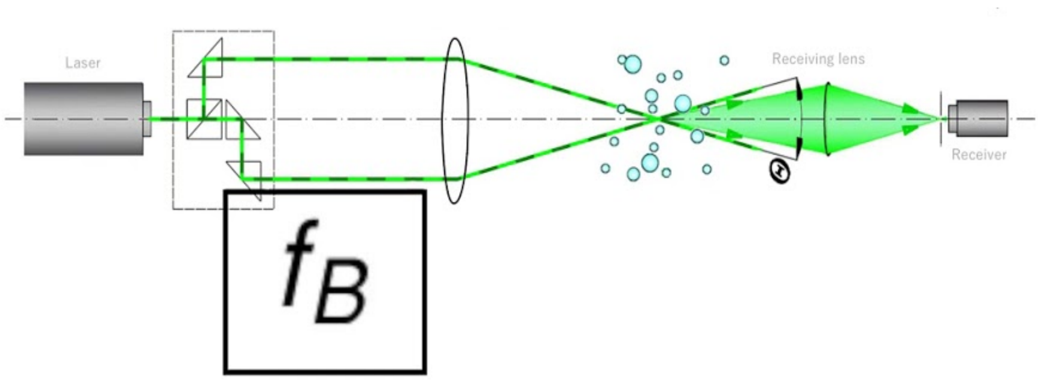
LDV: Laser Doppler Velocimeters
Invented in 1964, are used to detect contactlessly a wide range of velocities in moving fluids. More precisely, it's a technique used to measure the velocity of fluids carrying scattering particles (naturally existing or artificially seeded into the fluids).
The following table shows the possible cases of scattering, the corresponding coefficient and how the frequency is "scattered" in depdendency on direction.

Type	Coefficients	
Rayleigh ($r \ll \lambda$)	$\alpha_s \propto \frac{r}{\lambda^4}$	$f(\theta)$ constant with the angle
Mie ($r \approx \lambda$)	$\alpha_s \approx \lambda - const.$	$f(\theta)$ max for $\theta = 0$

The measured signal can be seen as due to:

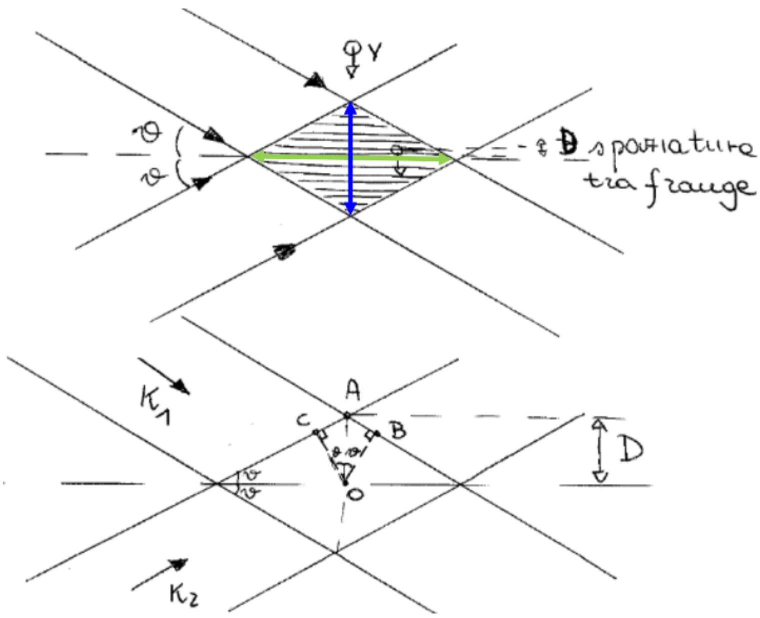
- Doppler effect
- Fringes crossing
- Interferometric phase shift

The set up



We are looking at a space-dependent interferometer. The two beams crosses at an angle equals to 2θ ($2\theta = \Theta$) and the distance between the beams (before the focusing lens) is such that $\dfrac{R}{f} \ll 1$.
The two collimated beams impenges off-axis on the lens, they get focused at distance f from the lens, calles "interferometric region".

LDV: interferometric fringes



Wavefronts:	approximately plane $\rightarrow \pm w_0$
Interaction zone:	$\Delta X = \pm 2 w_0 \cos\theta \qquad \Delta Y = \pm 2 w_0 \sin\theta$
Approximations:	typically $\Delta X \gg \Delta Y$ and θ is very small
Phase difference at any point:	$\Delta\Phi = \Phi_2 - \Phi_1 = k_2 s_2 - k_1 s_1$

Always looking at the picture, let's compute how the phase difference changes from O to A:

$$\Delta\Phi_{O\rightarrow A} = [\Phi_2 - \Phi_1]_{(O)} - [\Phi_2 - \Phi_1]_{(A)}$$

We can understand that, being D the fringe spacing, at the heighs of O and A on the photodetector there are two friges. Hence $\Delta\Phi_{O\to A} = 2\pi$. Now, pay attention to the fact that moving from O to C in the second beam means changing position without changing the phase, analogously from O to B for the beam 1 (it's not well drawn, but $\widehat{OCA}$ and $\widehat{OBA}$ are square corners).
Hence, from O to A:

$$\Delta\Phi_{2(O\to A)} = \Phi_{2(O\to C)} + \Phi_{2(C\to A)} = 0 + k_2\overline{CA} = +(2\pi/\lambda)D\sin\theta$$
$$\Delta\Phi_{1(O\to A)} = \Phi_{1(O\to B)} + \Phi_{1(B\to A)} = 0 - k_1\overline{BA} = -(2\pi/\lambda)D\sin\theta$$

$$\Delta\Phi_{O\to A} = \Delta\Phi_{2(O\to A)} - \Delta\Phi_{1(O\to A)} = 2\cdot(2\pi/\lambda)D\sin\theta = 2\pi \Rightarrow D = \frac{\lambda}{2\sin\theta}$$

<u>Relation between velocity and frequency</u> Thickness of the crossing region varies with position but D remains the same. To avoid misunderstandings, "D" is used as subscript to indicate the period and frequency of the diffused signal.
On the time axis, the period of the peaks depends on the velocity of the scattering particle(s), hence $T_D = \frac{D}{v}$.
In general $T_D \propto v^{-1}$, and $f_D = 1/T_D$.

$$\Rightarrow f_D = \frac{1}{T_D} = \frac{v}{D} = \frac{2\sin\theta}{\lambda}v \propto v$$